Math 256 Fall 2007
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Remarks on integrating factors, §2.1, 2.4.

Review of the method.

The method of integrating factors reduces the solution of the general scalar first order linear equation

$$y' \;+\; p(t)\,y \;=\; g(t), \tag{1}$$

to two integrations in time. Here $p(t)$ and $g(t)$ are given continuous functions on a time interval $\alpha \le t \le \beta$.

The method relies on the product rule,

$$\frac{d}{dt}\Big(e^{A(t)}\,y\Big) \;=\; e^{A(t)}\Big(y' + A'\,y\Big)\,.$$

Therefore, if $A(t)$ is chosen so that

$$\frac{dA}{dt} \;=\; p(t), \tag{2}$$

then solutions of (1) satisfy

$$y' + A'(t)\,y \;=\; y' + p(t)\,y \;=\; g(t),$$

so

$$\frac{d}{dt}\Big(e^{A(t)}\,y\Big) \;=\; e^{A(t)}\,g(t)\,. \tag{3}$$

Equation (2) is equivalent to the text's equation for $\ln\mu$ between equations (29) and (30). Equation (3) is equivalent to (32) in the text.

The analysis of y is reduced to two steps. **1.** Find an antiderivative $A(t)$ of $p(t)$ as in (2). **2.** Find an antiderivative, $k(t)$ of $e^{A(t)}\,g(t)$. Then

$$e^{A(t)}\,y(t) \;=\; k(t) \;+\; C\,, \qquad y(t) \;=\; e^{-A(t)}\Big(k(t) \;+\; C\Big)\,. \tag{4}$$

Examples where both antiderivatives are elementary are rare, so the practice examples in the text are of necessity repetitive. In class we worked the example on page 37. We found

$$e^{A(t)} = t^2, \qquad k(t) = t^4\,, \qquad \text{and} \qquad y \;=\; \frac{1}{t^2}\Big(t^4 + C\Big)\,.$$

Even when the primitives are not given by simple equations, the formulas yield information about y. We give two examples.

Solution of initial value problems, Theorem 2.4.1

To solve the initial value problem for (1) with

$$y(t_0) = y_0,$$

with t_0 a point of the interval, take

$$A(t) \;=\; \int_{t_0}^{t} p(s)\, ds. \tag{5}$$

Then A satisfies (2) and $A(t_0) = 0$, so

$$e^{A(t)}\, y(t)\big|_{t=t_0} \;=\; e^{A(t_0)}\, y(t_0) \;=\; e^0\, y_0 \;=\; y_0.$$

The Fundamental Theorem of Calculus yields

$$e^{A(t)} y(t) - y_0 \;=\; e^{A(t)}\, y(t)\big|_{t=t_0}^{t} \;=\; \int_{t_0}^{t} \frac{d}{dt}\Big(e^{A(t)}\, y(t)\Big)\, dt \;=\; \int_{t_0}^{t} e^{A(t)}\, g(t)\, dt\,,$$

where (3) is used in the last equality. Solving for y, this uniquely determines a continuously differentiable function

$$y(t) \;=\; e^{-A(t)}\Big(y_0 \;+\; \int_{t_0}^{t} e^{A(s)}\, g(s)\, ds\Big)\,. \tag{6}$$

Reversing the steps of the argument shows that this function $y(t)$ solves the initial value problem. This proves Theorem 2.4.1 on page 68 at the same time giving formula (5), (6) for the unique solution.

A conserved quantity, §2.6.

Define the function

$$\Psi(t, y) \;:=\; e^{A(t)}\, y \;-\; k(t)\,.$$

The formula for the general solution of (1) shows that if $y(t)$ is a solution of (1), then $\Psi(t, y(t))$ is a constant, C. It does not depend on time. For the example on page 37,

$$\Psi(t, y) \;=\; t^2\, y - t^4\,.$$

In the (t, y)-plane where direction fields are sketched, each solution curve of the ordinary differential equation lies on a level set $\Psi = C$. One way to find solution curves is to find the level curves of Ψ. For example, this can be done approximately using the MATLAB command `contour`. **Exercise.** Use `contour` on one hand and `dfield` on the other to solve the equation on page 37. They should agree.

Functions which are constant on integral curves of an ordinary differential equation are called *integrals of motion*. Quantities like energy or momentum often provide integrals of motion and are very important tools. In mechanics you may have encountered problems where the essential features can be easiy derived using such laws. A classic example is the computation of the escape velocity using only energy conservation (reference: Feynman Lectures, vol. 1).