## The Wronskian Theorems

### §1. Second order equations.

**Second Order Wronskian Theorem.** *Suppose that $y_1(t)$ and $y_2(t)$ are solutions of the seond order linear homogeneous equation $L\,y = 0$ on an interval, $I$. Then, the following are equaivalent.*

**1.** *For some $t_0 \in I$,*

$$\det \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0.$$

**2.** *For any values $a, b$ there are uniquely determined constants $c_1, c_2$ so that $y = c_1 y_1 + c_2 y_2$ is the solution of,*

$$L\,y = 0, \qquad y(t_0) = a, \quad y'(t_0) = b\,.$$

**3.** *Every solution of $Ly = 0$ is of the form $y = c_1 y_1 + c_2 y_2$ for real constants $c_1, c_2$. That is, $c_1 y_1 + c_2 y_2$ is the general solution.*

**4.** *(Theorem 3.3.1) For all $t \in I$,*

$$\det \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0.$$

**5.** *The solutions $y_1$ and $y_2$ are linearly independent.*

**Remarks. i.** The determinant appearing in **1** and **4** is called the **wronskian**.

**ii.** The equivalence of the first two is Theorem 3.2.2.

**iii.** The equivalence of the third with the first two is Theorem 3.2.4.

**iv.** The equivalence of the fourth with the first three is Theorem 3.3.1.

**v.** The equivalence of the fifth with the first four is Theorem 3.3.3.

**§2. Equations of order $n$.**

Suppose that

$$L \;=\; \frac{d^n}{dt^n} \;+\; a_{n-1}(t)\,\frac{d^{n-1}}{dt^{n-1}} \;+\; a_{n-2}(t)\,\frac{d^{n-2}}{dt^{n-2}} \;+\; \cdots \;+\; a_1(t)\,\frac{d}{dt} \;+\; a_0(t)\,,$$

is an $n^{\rm th}$ order linear differential operators with continuous real valued coefficients on an interval $I$.

Suppose that for $j = 1, 2, \dots, n$, $y_j(t)$ satisfies $L\,y_j = 0$.

**Definition.** The **wronskian** of these $n$ solutions is defined as the $n \times n$ determinant,

$$W(t) \;:=\; \det \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \cdots & & & \cdots \\ \cdots & & & \cdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}\,.$$

**Definition.** The solutions are **linearly dependent** when there are constants $a_1, a_2, \dots, a_n$ *not all zero* so that $a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0$. Otherwise they are **linearly independent**.

**General Wronskian Theorem.** *The following are equaivalent.*

**1.** *For some $t_0 \in I$, $W(t_0) \neq 0$.*

**2.** *For any values $b_0, b_1, \dots, b_{n-1}$ there are uniquely determined constants $c_j$ so that $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the solution of the initial value problem,*

$$L\,y \;=\; 0, \qquad y(t_0) = b_0, \;\; y'(t_0) = b_1, \;\; \cdots \;\; y^{(n-1)}(t_0) = b_{n-1}\,.$$

**3.** *Every solution of $Ly = 0$ is of the form $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ for real constants $c_j$. That is, $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the general solution.*

**4.** *For all $t \in I$, $W(t) \neq 0$.*

**5.** The solutions $y_1, y_2, \dots, y_n$ are linearly independent.

The proof is almost identical to the proof in the special case $n = 2$.

**§3. First order Systems.**

Suppose that

$$L \;=\; \frac{d}{dt} \;+\; A(t), \qquad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & & & a_{2n}(t) \\ & & & \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix},$$

is an first order linear system with continuous coefficients on an interval $I$.

Suppose that for $j = 1, 2, \dots, n$, $X^{(j)} = \big(x_1^{(j)}(t), x_2^{(j)}(t), \dots, x_n^{(j)}(t)\big)$ satisfies $L\,X = 0$.

**Definition.** The **wronskian** of these $n$ solutions is defined as,

$$W(t) \;:=\; \det\left[X^{(1)}(t), X^{(1)}(t), \dots, X^{(1)}(t)\right],$$

where the $n \times n$ matrix on the right has $j^{\text{th}}$ column equal to $X^{(j)}(t)$.

The wronskian is nonzero at $t_0$ if and only if the vectors $X^{(1)}(t_0), \dots, X^{(n)}(t_0)$ are linearly independent, if and only if the vectors $X^{(1)}(t_0), \dots, X^{(n)}(t_0)$ form a basis of $\mathbf{R}^n$ or $\mathbf{C}^n$ depending on the case.

**General Wronskian Theorem.** *The following are equaivalent.*

**1.** *For some $t_0 \in I$, $W(t_0) \neq 0$.*

**2.** *For any values $B \in \mathbf{R}^n$ there are uniquely determined constants $c_j$ so that $X = c_1 X^{(1)} + c_2 X^{(2)} + \cdots + c_n X^{(n)}$ is the solution of the initial value problem,*

$$L\,X \;=\; 0, \qquad X(0) = B\,.$$

**3.** *Every solution of $LX = 0$ is of the form $X = c_1 X^{(1)} + c_2 X^{(2)} + \cdots + c_n X^{(n)}$ for real constants $c_j$. That is, $X = c_1 X^{(1)} + c_2 X^{(2)} + \cdots + c_n X^{(n)}$ is the general solution.*

**4.** *For all $t \in I$, $W(t) \neq 0$.*

**5.** The solutions $X^{(1)}, X^{(2)}, \cdots, X^{(n)}$ are linearly independent.

The proof is almost identical to the proof in the special case $n = 2$.