Math 555 Final Exam December 19, 2001   Name ....

Instructions. 1. Two sides of an 8.5in. × 11in. sheet of notes from home. Closed book.
2. Show work and explain clearly.
3. 100 points total
4. There are 12 questions and 13 pages. They are mostly short.

1. (7=4+3 points) The analytic function $f(z)$ satisfies

$$f(1 + i) = 2, \qquad \text{and} \qquad f(.9 + 1.2i) = 2.3.$$

a. Estimate the value of f' at $z = 1 + i$.

b. Predict the value of $f(.9 + i)$

	1	2	3	4	5	6	7	8	9	10	11	12

2. (6 points) At what points z does df/dz exist when $f(z) = \bar{z}^2 = (x - iy)^2$?

3. (6 points) Mark each of the following assertions with a T if it is true. For the statement(s) which are false, provide a counterexample.

In each of the assertions $f = u + iv$ with u and v real valued twice continuously differentiable functions on the disk $\mathbf{D} := \{|z| < 1\}$.

a. If $f = u + iv$ is analytic in $\mathbf{D}$, then u and v are harmonic in $\mathbf{D}$.

b. If $u(x, y)$ and $v(x, y)$ are harmonic functions, then $f = u + iv$ is analytic in $\mathbf{D}$.

c. If $u + iv$ analytic in $\mathbf{D}$, then u and v satisfy the Cauchy-Riemann equations in $\mathbf{D}$.

d. If u and v satisfy the Cauchy-Riemann equations in $\mathbf{D}$, then $f = u + iv$ is analytic in $\mathbf{D}$.

4a. (9=5+4 points) The function $1/\sin z$ has a pole at $z = 0$. Compute the first two nonzero terms of the Laurent expansion valid near $z = 0$.

b. Find the largest value $R > 0$ so that the Laurent expansion converges for $0 < |z| < R$.

5. (5 points) Which of the following Fourier series is the expansion of a 2π periodic function which has an analytic continuation to a nonempty strip $|\operatorname{Im} z| < a$.

$$\textbf{a.} \ \sum_{n>0}^{\infty} n^3 e^{-\sqrt{n}} e^{inx}, \qquad \textbf{b.} \ \sum_{n>0}^{\infty} n^3 e^{-n^2} e^{inx}, \qquad \textbf{c.} \ \sum_{n>0}^{\infty} \frac{1}{n^4} e^{inx}.$$
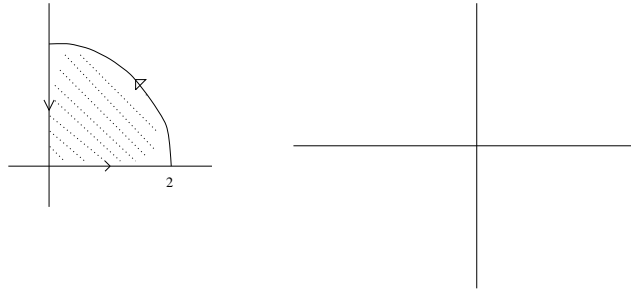
State the criterion you are applying.

6. (5 points) Which of the following functions has a singularity which is not isolated

$$\text{a. } \frac{1}{\sin z}\,, \qquad \text{b. } \frac{1}{\sqrt{z}}\,, \qquad \text{c. } \log z\,.$$

Explain briefly how you know.

7a. (12=4+4+4 points) On the axes provided, sketch the image of the circular quadrant on the left by the function $f(z) = \log z$, where the logarithm is real for positive real values of z. Indicate important coordinates.

b. On the axes provided, sketch the image of the circular quadrant on the left by the function $f(z) = 1/z$. Indicate important coordinates.
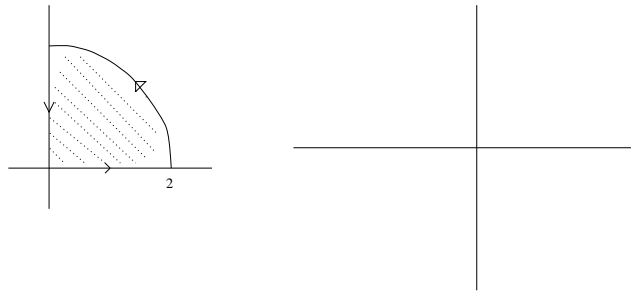
c. On the axes provided, sketch the image of the rectangle in the figure below by the function $h(z) = e^z$. Indicate important coordinates.
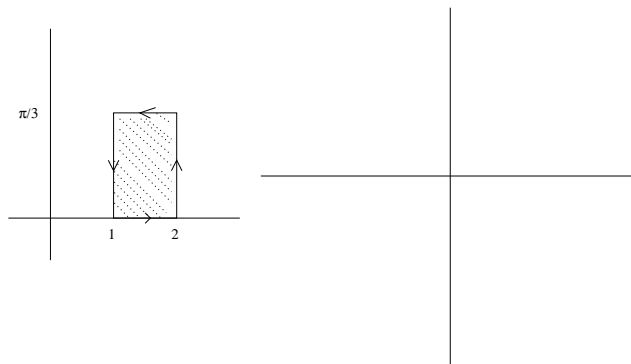
8. (6 points) Let $k(z) = z(z-1)(z-2)^2 e^z$. How many times does the image $k(C)$ of the circle $|z| = 1.5$ wind around the origin? How do you know?

9. (12 points) Find the exact value of

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \, dx \, .$$

10. (13 points) Find the exact value of

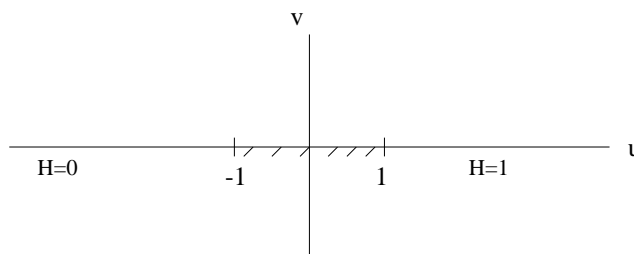$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} \, dx \, .$$

11. (6 points) (308/7) In problem 308/6 you computed a steady temperature $H(u, v)$ distribution in the half plane $v > 0$. The harmonic function H is bounded and satisfies

$$H(u, 0) = 0 \ \text{ for } \ u < -1, \qquad \frac{\partial H(u, 0)}{\partial n} = 0 \ \text{ for } \ -1 < u < 1, \qquad H(u, 0) = 1 \ \text{ for } \ u > 1.$$

The solution was found to be

$$H(u, v) = \frac{1}{2} + \frac{1}{\pi} \, \text{Re}[\arcsin(u + iv)]$$

The segment $-1 < u < 1$ on the x-axis is insulated. The problem is indicated in the figure below.


Find the steady state temperature $T(x, y)$ in the quadrant $x > 0, y > 0$ when the boundary segment $y > 1$ on the y-axis is kept at $T = 0$, the temperature on the part of the x-axis with $x > 1$ is kept at $T = 1$, and the remaining boundary is insulated. The problem is indicated in the figure below.
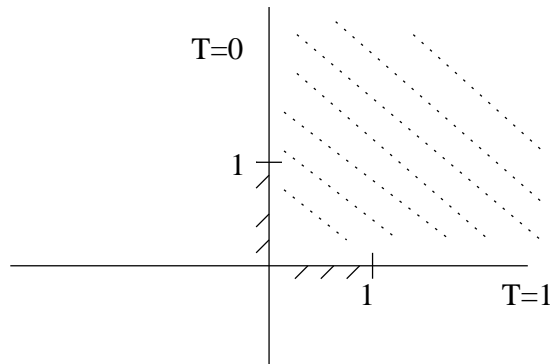
In addition to the indicated boundary conditions, the desired harmonic function $T(x, y)$ must be bounded.

12. (12 points) This problem takes two pages. The function $F(z) = z + 1/z$ defines a conformal mapping of that part of the upper half plane with $|z| > 1$ onto the upper half plane. In addition $F \approx z$ for $|z| \to \infty$ (Figure 17 in the text).
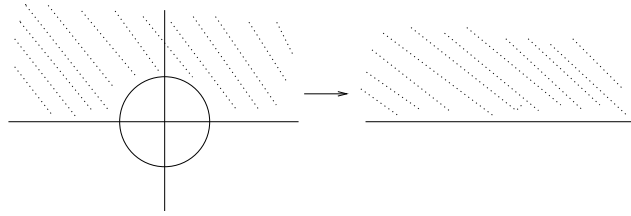
This mapping can be used to find the steady state incompressible, irrotational inviscid fluid flow past the semicircular obstacle on the left. The velocity is tangent to the boundary curves and is required to converge to $(1, 0)$ as $|x, y| \to \infty$.

a. Find the steady state velocity at the point $1 + i$.

b. Find the equation of the streamline (=particle path) which is asymptotic to the line $y = 1$ as $|x| \to \infty$.

c. Find the point on the y-axis which belongs to the streamline from part b. Ans. $(0, (1+\sqrt{5})/2)$.

Discussion. The fluid layer which is one unit deep at $x = -\infty$ squeezes by the obstacle in a layer $(\sqrt{5}/2) - 1$ thick. The fluid must travel faster to get by in this thinner layer. The faster speed makes for lower pressure and therefore lift on the roof of the semicircle.