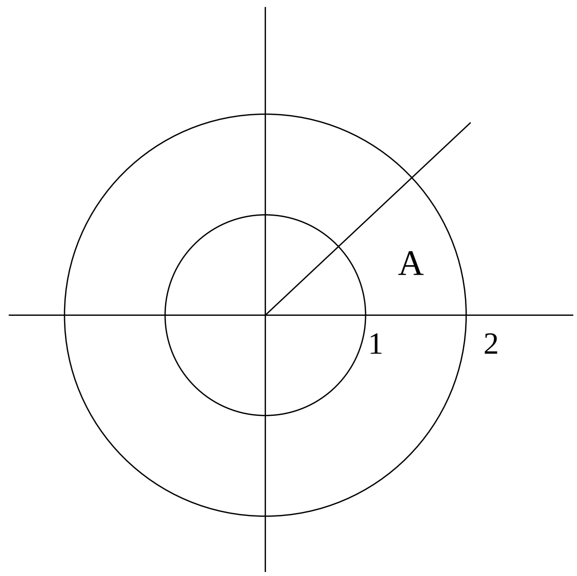
Math 555, Fall 2002 Prof. J. Rauch

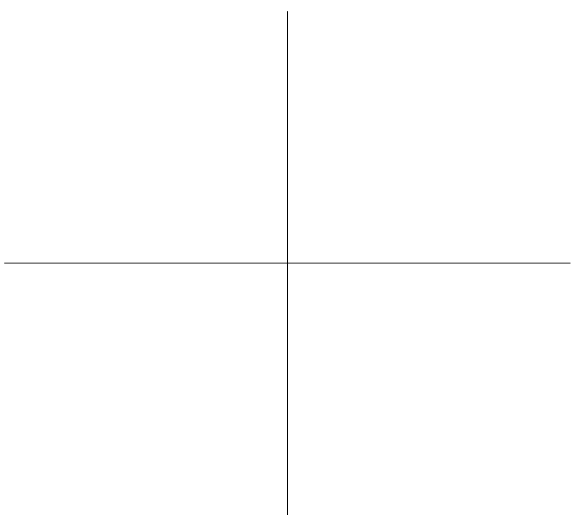
Midterm Exam October 17, 2002

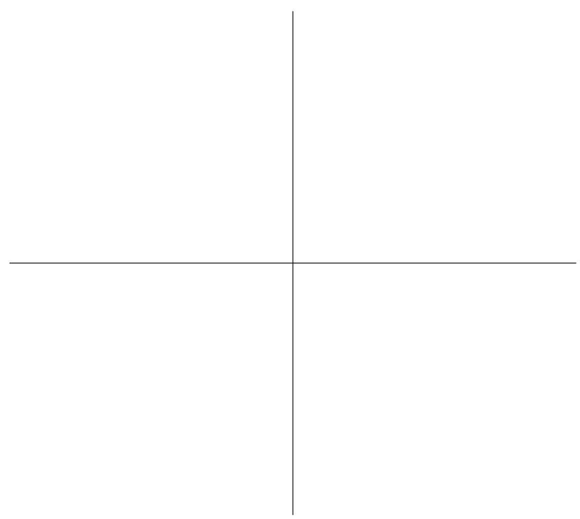
Instructions. 1. Two sides of a 3.5in. $\times$ 5in. sheet of notes from home. Closed book.
2. Show work and explain clearly.
3. There are six questions, one per page. 100 points total

1. (15 points) Show that if $f(z)$ is an analytic function on a connected open set Ω and f depends only on x where $z = x + iy$, then f is a constant.

2. (14 points) Suppose that $f(z) = 1/z$, $g(z) = z^2$ and $A = \{z : 1 < |z| < 2,\ \ 0 < \arg z < \pi/4\}$. The set A is sketched below. Sketch the regions $f(A) = \{f(z) : z \in A\}$ and $g(A) = \{g(z) : z \in A\}$ on the axes provided. Indicate important lengths and angles in the sketch.







3. (8+6 points) **a.** At what point(s) of the closed disc with center at i and radius equal to 2 does the modulus of the analytic function $f(z) = z^4$ attain its maximum and minimum values.

b. Explain why this is consistent with the maximum and minimum modulus principles for analytic functions.

4. (10+10+10 points) Using theorems of complex analysis, evaluate the following three integrals. Be sure to state the general formula you are applying and to what function(s).

a.
$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{1}{(z-2)\,(z-4)^3}\,dz$$

b.
$$\frac{1}{2\pi i}\oint_{\partial\Omega}\frac{1}{(z-2)\,(z-4)^3}\,dz\,,$$

where Ω is the annular region $1 \leq |z| \leq 3$. As usual, $\partial\Omega$ denotes the boundary of Ω oriented in the standard sense.

c.
$$\frac{1}{2\pi i}\oint_{\partial R}\frac{1}{(z-2)\,(z-4)^3}\,dz\,,$$

where R is the annular region $3 \leq |z| \leq 5$.

5. (5+5+5 points) Liouville's Theorem asserts that an analytic function which is defined on the entire complex plane and is bounded must be a constant function. Does this result imply that any of the following functions is constant? Explain.

a. $\dfrac{1}{1+x^2+y^2}$, **b.** $\dfrac{1}{z^2}$, **c.** $\sin z$.

6. (6+6 points) **a.** The equation $z(t) = 6\,e^{it}$, $0 \le t \le 1$ is a parametric equation a curve. Sketch it. Be sure to label important points, coordinates, angles, distances, ... etc.

b. Sketch a domain in the complex z plane which is mapped in a one to one way onto the upper half plane $\{w : \operatorname{Im} w > 0\}$ by the mapping $w = e^z$.