Math 555, Fall 2009 Prof. J. Rauch

Midterm Exam October 22, 2009

Instructions. 1. Closed book. Two sides of a 3.5in. × 5in. sheet of notes from home.
2. No electronics, phones, cameras, . . . etc.
3. Show work and explain clearly.
4. There are 7 questions. They consist of 14 short subquestions each worth 5 points.
70 points total. You have about 5.5 minutes per short question. Be efficient.

1. (5 points). Is there a disk with postive radius on which the function $f(z) = x^2 - iy^2$ is analytic?

Solution. The function f has continuous partial derivatives of all orders. To check analyticity it suffices to check whether the Cauchy-Riemann equations $\partial f/\partial x = (1/i)\partial f/\partial y$ are satisfied. Compute

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = -2iy, \qquad \frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y} = 2(x+y)\,.$$

Thus the Cauchy-Riemann equations are satisfied only on the line $x = -y$ and therefore **not** on any disk of postive radius since all such disks contain points off this line.

Remarks. i. The *definition* of analytic requires C-R on an open set. If you said that f was analytic on $\{x = -y\}$ that is FALSE. Analytic on this line *means* that C-R is satisifed on an open set containing the line. I did not deduct points for this error. I worded the problem exactly so that you would not fall into the trap.

ii. Some observed that u and v satisfy Laplace's equation at no point. That is a shorter solution.

-1 point for not observing that disks cannot be contained in the line.

Prob.	Score
1	
2	
3	
4	
5	
6	
7	
Sum, %	

2. (5+5 points). Let Γ denote the contour connecting $(0,0)$ to $(1,1)$ along the curve $y = x^2$. **a.** Using the definition of line integral evaluate,

$$\int_\Gamma z \, dz\,.$$

b. Evaluate the same integral a second way using an antiderivative of z and the fundamental theorem of calculus.

Solution. Parameterize the curve as (t, t^2), $0 \le t \le 1$. The definition of the line integral is then

$$\begin{aligned}
\int_0^1 (x(t) + iy(t))(dx/dt + idy/dt)\, dt &= \int_0^1 (t + it^2)(1 + 2it)\, dt \\
&= \int_0^1 (t + 3it^2 - 2t^3)\, dt \\
&= \left(\frac{t^2}{2} + it^3 - \frac{t^4}{2}\right)_0^1 = 1/2 + i - 1/2 = i.
\end{aligned}$$

For part **b**,

$$\int \frac{d(z^2/2)}{dz}\, dz = \left.\frac{z^2}{2}\right|_0^{1+i} = \frac{(1+i)^2}{2} - \frac{0^2}{2} = \frac{1 + 2i - 1}{2} = i\,.$$

3. (5+5+5 points). **a.** For $\mathbb{C} \ni a \neq 0$ and $1 \leq n \in \mathbb{Z}$ describe and sketch the curve swept out by the function $f(z) = 1 + a\, z^n$ as z turns once counterclockwise around the unit circle $|z| = 1$ beginning at the point $z = 1$. Label axes, starting point, ending point, and critical dimensions.

b. Find the point(s) z on the unit circle where $|f(z)|$ is largest.

c. What does this bound on $|f|$ tell you about the values of $f(z)$ in the disk $|z| \leq 1$. What result are you using?

Solution. The image is the circle centered at $(1, 0)$ and radius $|a|$. The image is swept out in the postive sense, that is clockwise, starting and ending at the point $1 + a$. The image makes n turns about the circle at constant speed.

The largest value of $|f(z)|$ occurs when this circle passes through the point $1 + |a|$ on the x-axis. This occurs at the points z on the unit circle with $z^n = a^*/|a|$. For these,

$$a\, z^n \;=\; a\, \frac{a^*}{|a|} \;=\; \frac{|a|^2}{|a|} \;=\; |a|\,.$$

These z are the n^{th} roots of $a^*/|a|$ so lie at the vertices of a regular n-gon inscribed in the unit circle with one of its vertices at $a/|a|$.

Since $|f(z)| \leq 1 + |a|$ for $|z| = 1$, the Maximum Modulus Theorem applied to the analytic function f on the disk $|z| \leq 1$ implies that,

$$|f(z)| \leq 1 + |a|, \qquad \text{for}, \qquad |z| \leq 1.$$

Remarks on grading.

a. -1 for no direction. -1 for no n turns. -1 for wrong beginning/end. -1 for drawing assuming $a > 0$ real.

4. (5+5+5 points). Let $\mathbb{A}$ denote the annulus $0.5 \le |z| \le 4$. Use Theorems of Cauchy to evaluate each of the following integrals. State the general formula that you are using.

$$\text{a. } \oint_{\partial\mathbb{A}} \sin(z^2)\, dz, \qquad \text{b. } \oint_{\partial\mathbb{A}} \frac{\sin(z^2)}{z-i}\, dz, \qquad \text{c. } \oint_{\partial\mathbb{A}} \frac{\sin(z^2)}{(z-i)^2}\, dz\,.$$

Solution. a. Let $f(z) = \sin(z^2)$ an entire function. The basic Cauchy Theorem about the integral over a boundary of a function analytic throughout the domain shows that the first integral vanishes.

b,c. The second and third integrands are singular at the interior point i. They can both be evaluated using the Residue Theorem. They both can be evaluated using the Cauchy Integral Formula,

$$\frac{n!}{2\pi i} \int_{\partial\mathbb{A}} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \;=\; f^{(n)}(z)\,.$$

Using the latter, yields the answers,

b.

$$2\pi i\, \sin(i^2) = 2\pi i \sin(-1),$$

c.

$$2\pi i\, \frac{d\sin(z^2)}{dz}\bigg|_{z=i} = 2\pi i\, \cos(z^2)\, 2z\big|_{z=i} = 2\pi i\, \cos(-1)\, 2i = -4\,\pi\, \cos(-1)\,.$$

5. (5+5 points). The function $f(z)$ is analytic on $\mathbb{C} \setminus \{1+i\,, 2\}$, that is, with two isolated singularities. A Taylor series of f is,

$$f(z) \ = \ 1 + 3z + z^2 + \ \cdots \,.$$

a. Find the largest open disk on which the Taylor series converges.

b. The inverse function theorem guarantees that f is invertible on a neighborhood of $z = 0$. How do you know this from the Taylor series?

Solution. a. From the form of the terms one sees that it is a power series centered at $z_0 = 0$. The series is convergent on the largest disk $|z| < R$ on which f is analytic. The singularity closest to the origin is $1+i$ so the largest disk is

$$|z| \ < \ |1+i| \ = \ \sqrt{2}\,.$$

b. The Inverse Function Theorem guarantees that the function is invertible on a neighborhood of $z = 0$ when $f'(0) \neq 0$. From the power series compute $f'(0) = 3 \neq 0$.

Some people offered the following argument. For z small $f(z) \approx 1 + 3z$ and the map $w = 1 + 3z$ is invertible with inverse $z = (w-1)/3$. Though this does not cite the inverse function theorem it does have the essence of the phenomenon and was graded as worth 3 points.

A distressing number of people confused invertibility in the sense of the Inverse Function Theorem with invertibility in the the sense of $1/f(z)$. No partial credit was offered for this.

6. (5+5 points). $k(z)$ is analytic in the annulus $1 < |z| < 2$.

a. Use one of the basic theorems of this course to show that if $1 < \rho_1 < \rho_2 < 2$, then

$$\frac{1}{2\pi i} \oint_{|z|=\rho} k(z)\, z^4 \, dz$$

has the same value for $\rho = \rho_1$ and $\rho = \rho_2$.

b. This integral is equal to which coefficient in the Laurent expansion,

$$k(z) \;=\; \sum_{n=-\infty}^{\infty} c_n\, z^n\,.$$

Solution. a. Denote by C_ρ the circle of radius ρ in the positive sense. The oriented boundary of the annulus $R = \{\rho_1 < |z| < \rho_2\}$ is

$$\partial R \;=\; C_{\rho_2} \;-\; C_{\rho_1}\,.$$

Since $f(z) := z^4\, k(z)$ is analytic throughout R and on ∂R, Cauchy's Theorem implies that

$$\int_{\partial R} f(z)\; dz \;=\; 0\,.$$

Since

$$\int_{\partial R} f(z)\; dz \;=\; \int_{C_{\rho_2}} f(z)\; dz \;-\; \int_{C_{\rho_1}} f(z)\; dz,$$

one has

$$\int_{C_{\rho_2}} f(z)\; dz \;=\; \int_{C_{\rho_1}} f(z)\; dz,$$

proving the desired result. Some used the principal of deformation of contours (for which the above is the proof).

b. The coefficient of z^{-5} which is c_{-5}. To see this write,

$$z^4\, k(z) \;=\; \sum_{n=-\infty}^{\infty} c_n z^{n+4}.$$

Since the sum is uniformly convergent on $|z| = \rho$ one can interchange sum and integral. And,

$$\oint_{|z|=\rho} z^{n+4}\; dz = \begin{cases} 0 & \text{when } n+4 \neq -1 \\ 2\pi i & \text{when } n+4 = -1 \end{cases}\,.$$

7. (5 points). The function, $g(z)$ is analytic in $|z| < 2$ with,

$$|g(z)| \leq M, \qquad \text{for}, \qquad |z| \leq 1\,.$$

Find an upper bound for $|g'(1/2)|$.

Solution. On the disk $|z - 1/2| \leq 1/2 := R$ the function g is analytic with $|g| \leq M$. This is the largest disk with center $z_0 = 1/2$ on which you are given the bound $|g| \leq M$. Any larger disk extends outside $|z| \leq 1$. Cauchy's inequality,

$$|g^{(n)}(\text{center})| \leq \frac{n!\, M_R}{R},$$

with $n = 1$, $R = 1/2$, $z_0 = 1/2$, yields,

$$|g'(1/2)| \leq \frac{M}{R} = \frac{M}{1/2} = 2M\,.$$

+1 for mentioning Cauchy inequality.