Math 555, Fall 2009 NAME ....

Final Exam Solutions and Grading Scheme**. December 20, 2009**

Instructions. 1. Closed book. Four sides of a 3.5in. × 5in. sheet of notes from home.
2. No electronics, phones, cameras, . . . etc.
3. Show work and explain clearly.
4. There are 8 questions. *The last three problems are long!* 53 points total.

1. (2+2+2 points). **a.** Let Γ be the straight line segment connecting $z = 0$ to $z = 1 + i$. Compute

$$\int_\Gamma x\, dz\,.$$

b. For $|\zeta| < 1$ compute

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{\sin z}{(z-\zeta)^{n+1}}\ dz\,.$$

c. For $|\zeta| > 1$ compute the integral in part **b.**

Solution. **a.** Parameterize as $z(t) = t(1 + i) = x(t) + iy(t)$, $0 \le t \le 1$. Then $x(t) = t$ and $dz/dt = 1 + i$. The definition of line integral yields,

$$\int_0^1 x(t)\ \frac{dz(t)}{dt}\, dt \ = \ \int_0^1 t\,(1+i)\, dt \ = \ (1+i)\int_0^1 t\ dt \ = \ \frac{(1+i)}{2}\,.$$

b. The Cauchy Integral Formula shows that this is equal to

$$\frac{1}{n!}\ \frac{d^n(\sin z)}{dz^n}\bigg|_{z=\zeta}\ .$$

c. In this case the integrand is analytic in the disk and Cauchy's Theorem implies that the integral *is equal to zero.*

2. (2+2+2 points). The function $H(z)$ is analytic on the disk $\{|z| < \rho\}$ with $H(0) = 0$ and $H(0.01) = .02(1+i)$.

a. Compute an approximate value for $H'(0)$.

b. Compute an approximate value for $H(0.01i)$.

c. The function $\gamma(z)$ is analytic in $\{|z| < 1\}$ with Taylor expansion at the origin equal to

$$\gamma(z) \;=\; z^m \;+\; \text{higher order terms}, \qquad m \geq 2\,.$$

Describe in words the image by γ of a small circle $|z| = \rho << 1$.

Solution. a. An approximate value of the derivative is given by the difference quotient,

$$H'(0) \;\approx\; \frac{H(0.01) - H(0)}{0.01 - 0} \;=\; \frac{.02(1+i)}{.01} \;=\; 2(1+i)\,.$$

b.

$$H(0.01i) \approx H(0) + 0.01\, i H'(0) \approx 0 + 2(1+i)(0.01) = .02(1+i)\,.$$

c. Parameterize the circle as $\rho\, e^{i\theta}$ then

$$\gamma \approx \rho^m\, e^{im\theta},$$

so the image is a contour that winds m times around the origin in the clockwise direction not deviating far from the circle of radius ρ^m traversed at constant angular velocity.

3. (3+3 points). **a.** Compute

$$\mathrm{Res}_{z=0}\ \frac{1}{z\ \sin z}\,.$$

b. Compute

$$\mathrm{Res}_{z=\pi}\ \frac{1}{z\ \sin z}\,.$$

Solution. a.

$$\frac{1}{\sin z}\ =\ \frac{1}{z - z^3/6 + \cdots}\ =\ \frac{1}{z}\,\frac{1}{1 - z^2/6 + \cdots}\ =\ \frac{1}{z}\left(1 + z^2/6 + \cdots\right).$$

So,

$$\frac{1}{z\ \sin z}\ =\ \frac{1}{z^2}\left(1 + z^2/6 + \cdots\right).$$

The residue at $z = 0$ is equal to zero.

b. At $z = \pi$, $\sin z$ vanishes with derivative equal to -1 so

$$\sin z = -(z-\pi) + \cdots\,, \qquad \frac{1}{z}\ =\ \frac{1}{\pi} + \cdots\,.$$

Therefore

$$\frac{1}{z\ \sin z}\ =\ \left(\frac{1}{\pi} + \cdots\right)\frac{1}{-(z-\pi) + \cdots}\ =\ \left(\frac{1}{\pi} + \cdots\right)\frac{-1}{z-\pi}\,\frac{1}{1 + \cdots}$$

The residue at $z = \pi$ is equal to $-1/\pi$.

4. (3+2 points). **a.** Find three nonzero terms of the Laurent expansion of the function

$$g(z) \;=\; \frac{1}{z-\zeta}, \qquad \zeta \neq 0,$$

in the region $|z| > |\zeta|$.

b. The function $h(z)$ is analytic in the punctured disk $0 < |z| < 1$ with Laurent expansion

$$h(z) \;=\; \sum_{n=-\infty}^{\infty} A_n\, z^n\,.$$

In addition one has

$$\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{h(z)}{z^3}\, dz \;=\; \sqrt{11}.$$

Find the value of one of the coefficients A_n.

Solution. a.

$$\frac{1}{z-\zeta} = \frac{1}{z(1-\zeta/z)} = \frac{1}{z}\,\frac{1}{1-\zeta/z}\,.$$

Since $|\zeta/z| < 1$ the last factor is a sum of a geometric series,

$$\frac{1}{z-\zeta} = \frac{1}{z}\Big(1 + \zeta/z + (\zeta/z)^2 + (\zeta/z)^3 + \cdots\Big)\,.$$

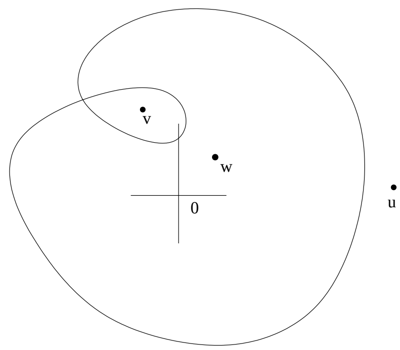
Ans.

$$\frac{1}{z} \;+\; \frac{\zeta}{z^2} \;+\; \frac{\zeta^2}{z^3} + \cdots\,.$$

b. Writing h as a sum, and therefore the integral as a sum (the series converges uniformly on the circle $|z| = 1/2$) all the summand integrals vanish (they are integrals of derivatives over a closed curve) except one to yield

$$\sqrt{11} \;=\; \frac{1}{2\pi i}\oint_{|z|=1/2} \frac{A_2\, z^2}{z^3}\, dz \;=\; A_2\,.$$

5. (3+1 points). A function $k(z)$ is analytic on a neighborhood of the closed disk $\{|z| \leq 1\}$. The image by k of the circle $|z| = 1$ is the following curve covered once.



a. How many roots does the equation $k(z) = u$, $k(z) = v$, and, $k(z) = w$ have in the unit disk $|z| \leq 1$?

b. How do you know that the curve is traversed in the positive (counter clockwise) direction as z goes around the unit circle in the positive sense.

Solution. a. The number of roots of $k(z) = u$ is the number of zeroes of $k(z) - u$. Since k has no poles the argument principal asserts that that number of zeroes is the number of times that $k(z) - u$ winds around the origin which is equal to the number of times that $k(z)$ winds around u which is **zero**.

The number of zeroes of $k(z) = v$ is the number of times the curve winds about v, equals **two**.

The number of zeroes of $k(z) = w$ is the number of times the curve winds about w, equals **one**.

b. The winding number around the origin is equal to the number of zeroes and is therefore nonnegative.

If k were meromorphic, then the winding number would be $Z - P$ and could be negative.

6. (8 points). Compute for positive real values of ω,

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{4+x^2}\, dx\,.$$

Solution. The integral is absolutely convergent since the modulus of the integrand is bounded by $(4+x^2)^{-1}$ which is absolutely integrable. The answer is the real part of

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{4+x^2}\, dx \;=\; \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\omega x}}{4+x^2}\, dx\,.$$

Let

$$f(z) \;=\; \frac{e^{i\omega z}}{4+z^2} \;=\; \frac{e^{i\omega z}}{(z+2i)(z-2i)} \;=\; \frac{e^{i\omega z}}{(z+2i)}\,\frac{1}{z-2i} \;:=\; g(z)\,\frac{1}{z-2i}\,. \qquad (0)$$

$f(z)$ has poles at $\pm 2i$.

The function $e^{i\omega z}$ decays exponentially as $\operatorname{Im} z \to +\infty$ (1 point).

Denote by C_R the upper half of the semicircle $|z| = R$ traversed in the counter clockwise sense. The residue theorem applied to the half disk capped by C_R mplies that for $R > 2$,

$$\int_{-R}^{R} \frac{e^{i\omega x}}{4+x^2}\, dx \;+\; \int_{C_R} \frac{e^{i\omega z}}{4+z^2}\, dz \;=\; 2\pi i\ \operatorname{Res}_{z=2i}\Big(\frac{e^{i\omega z}}{4+z^2}\Big)\,. \qquad \text{(3 points)} \qquad (1)$$

The factor $g(z)$ in (0) is analytic on a neighborhood of $2i$ so the pole is simple with

$$\operatorname{Res}_{z=2i}\Big(\frac{e^{i\omega z}}{4+z^2}\Big) \;=\; g(2i) \;=\; \frac{e^{i\omega(2i)}}{(2i+2i)} \;=\; \frac{e^{-2\omega}}{4i}\,. \qquad \text{(2 points)}$$

On the curve C_R,

$$|e^{i\omega z}| \le 1, \qquad \text{and} \qquad \left|\frac{1}{4+z^2}\right| \;\le\; \frac{1}{R^2-4}\,.$$

Thus the integral over C_R, whose length is πR, is no larger than

$$\frac{\pi R}{R^2-4} \;\to\; 0 \qquad \text{as} \quad R\to\infty\,. \qquad \text{(1 point)}$$

Passing to the limit $R\to\infty$ in (1) implies that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{4+x^2}\, dx \;=\; 2\pi i\,\frac{e^{-2\omega}}{4i} \;=\; \frac{\pi\, e^{-2\omega}}{2}\,. \qquad \text{(1 point)}$$

The answer is the real part of this number, hence $\pi e^{-2\omega}/2$.

When $\cos\omega z$ is used without noticing its exponential divergence on C_R, (5 points max.).

7. (8 points). Compute

$$\int_0^\infty \frac{\sqrt{x}}{4+x^2}\, dx\,.$$

Solution. For $R>2$ and $0<\epsilon<<1$, use the thermometer contour $C(\epsilon,R)$ which starts at the point on the circle $|z|=R$ with $y=i\epsilon$ and goes counter clockwise nearly all the way around the circle of radius R to the point with $y=-i\epsilon$. The contour then descends parallel to the x-axis to the point $-i\epsilon$. Then, goes around a circle of radius ϵ. Then mounts parallel to the x-axis to return to the starting point. (1 point)

Use the function $z^{1/2}=\sqrt{r}\, e^{i\theta/2}$ with $0<\theta<2\pi$. This function is analytic on $\mathbb{C}\setminus[0,\infty[$. The values of the square root for z just above the real axis converge to the positive square root as the imaginary part tends to zero. The values for z just below the real axis converge to the negative square root as the imaginary part increases to zero. The convergence is uniform on bounded sets of the positive real axis $[0\,\infty[$.

$C(\epsilon,R)$ encloses poles, at $\pm 2i$. The residue theorem yields,

$$\oint_{C(\epsilon,R)} \frac{z^{1/2}}{4+z^2}\, dz \;=\; 2\pi i\Big(\mathrm{Res}_{z=2i}\Big(\frac{z^{1/2}}{4+z^2}\Big) \;+\; \mathrm{Res}_{z=-2i}\Big(\frac{z^{1/2}}{4+z^2}\Big)\Big)\,. \qquad \text{(3 points)} \quad (1)$$

A computation as in problem 6, shows that the poles are simple and the residues are

$$\mathrm{Res}_{z=2i}\Big(\frac{z^{1/2}}{4+z^2}\Big) = \frac{(2i)^{1/2}}{2i+2i} = \frac{1+i}{4i}, \qquad (2)$$

$$\mathrm{Res}_{z=-2i}\Big(\frac{z^{1/2}}{4+z^2}\Big) = \frac{(-2i)^{1/2}}{-2i-2i} = \frac{1-i}{-4i}. \qquad \text{(2 points)} \qquad (3)$$

The continuity of the integrand at $z=0$ shows that as $\epsilon\to 0$ the integral over the circular arc of size ϵ tends to zero. The integral on the horizontal path above the axis converges to $\int_0^R \sqrt{x}\,(4+x^2)^{-1}\,dx$. For the horizontal path below the axis, the square root is mutliplied by -1 and the direction of the integral is reversed. Using the two factors of -1 the limit is also $\int_0^R \sqrt{x}\,(4+x^2)^{-1}\,dx$. Therefore,

$$\lim_{\epsilon\to 0}\oint_{C(\epsilon,R)} \frac{z^{1/2}}{4+z^2}\, dz \;=\; 2\int_0^R \frac{\sqrt{x}}{4+x^2}\,dx \;+\; \int_{|z|=R} \frac{z^{1/2}}{4+z^2}\,dz\,. \qquad \text{(1 point)} \qquad (4)$$

On the circle of radius R the integrand is no larger than $R^{1/2}/(R^2-4)$ so the integral over the curve of length $2\pi R$ is no larger than $2\pi R^{3/2}/(R^2-4)\to 0$ as $R\to\infty$. Therefore,

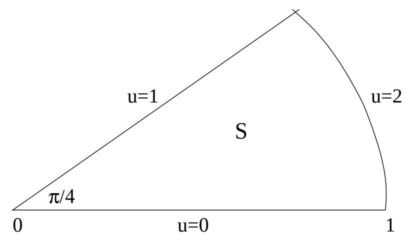
$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\oint_{C(\epsilon,R)} \frac{z^{1/2}}{4+z^2}\, dz \;=\; 2\lim_{R\to\infty}\int_0^R \frac{\sqrt{x}}{4+x^2}\,dx \;=\; 2\int_0^\infty \frac{\sqrt{x}}{4+x^2}\,dx\,. \qquad \text{(1 point)}$$

Combining yieilds

$$2\int_0^\infty \frac{\sqrt{x}}{4+x^2}\,dx \;=\; 2\pi i\Big(\frac{1+i}{4i}+\frac{1-i}{-4i}\Big) = \frac{2\pi i}{4i}\big((1+i)-(1-i)\big) \;=\; \pi\,.$$

Alternate Solution. Using the boundary of the upper half of the disk of radius R yields a slightly shorter evaluation.

8. (6+4 points). Denote by S the open sector $0 < |z| < 1,\ 0 < \arg z < \pi/4$, where the branch of the argument has values in $]-\pi, \pi[$.



a. Find a conformal mapping of S to the upper half plane, $\operatorname{Im} z > 0$.

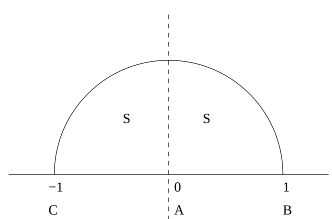
b. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad \text{in} \qquad S,$$

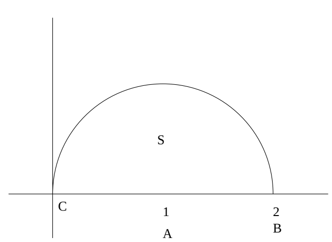
with boundary values as indicated in the figure. If you did not get part **a**, you may denote by $G(z)$ such a conformal map and continue from there.

Application. If u is a steady state temperature, the gradient of u gives the heat current. If v is the harmonic conjugate, also computed in **a**, then the level sets $v =$ constant give the lines of heat flow.

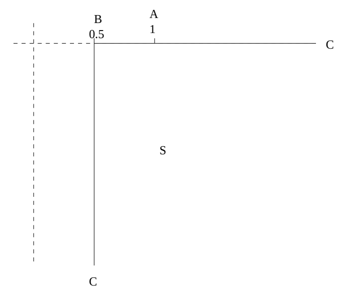
Solution. a. Denote the vertices $0, 1, e^{i\pi/4}$ as A, B, C. We will keep track of them throughout. The map $z \to z^4$ maps to the unit upper half disk with A, B, C going to $0, 1, -1$.



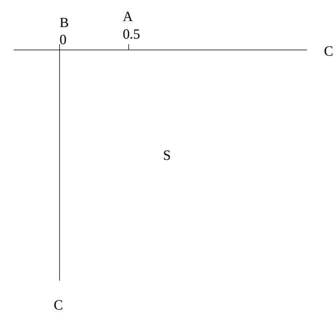
The map $z \to z + 1$ yields



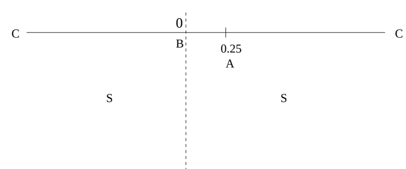
The map $z \to 1/z$ yields,



Next map $z \to z - 1/2$,



Then $z \to z^2$



The map $z \to -z$ completes the construction.

b. Denote by $F(z)$ that maps to the lower half plane in the last diagram. In S use the three harmonic functions

$$\theta_1 = arg(z - 0.25), \quad 0 \geq \theta_1 \geq -\pi, \qquad \theta_2 = arg(z), \quad 0 \geq \theta_2 \geq -\pi, \qquad \text{and,} \qquad 1\,.$$

Write $u = a \cdot 1 + b\theta_1 + c\theta_2$. The boundary condition $u = 1$ on AC yields

$$1 = a + b \cdot 0 + c \cdot 0, \qquad \text{so,} \qquad a = 1\,.$$

The boundary condition $u = 0$ on AC yields

$$0 = a + b \cdot 0 + c \cdot (-\pi), \qquad \text{so,} \qquad c = 1/\pi\,.$$

The boundary condition $u = 2$ on BC yields

$$2 = a + b(-\pi) + c(-\pi), \qquad \text{so,} \qquad b = -2/\pi\,.$$

This determines u. *The solution of the boundary value problem is equal to $u(F(z))$.*

Alternate endgame. If you forget the two angle strategy, you can solve Dirichlet problems with two boundary values using a dictionary using one angle and the constants. You can solve the problem with boundary values $0, 1, 2$ on the three segments by writing the values as the sum of $0, 1, 1$ and $0, 0, 1$ each of which has only two values.