Math 555, Fall 2011, Prof. J. Rauch NAME ....

Final Exam, December 19, 2011

Instructions. 1. Closed book. Two 3.5in. $\times$ 5in. sheet (four sides) of notes from home.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are 7 questions on 7 pages. 44 points total.

1. (2+2+1 points). **i.** Define

$$h(z) \; := \; \frac{1}{z(1-z^2)}\,.$$

Find the radius of convergence of the Taylor series of h about the point $2+i$.

ii. Find the Laurent expansion of $h(z)$ valid on the annulus $0 < |z| < 1$.

iii. $h(z)$ has a different Laurent expansion on a different annulus centered at the origin. Find that annulus. *You need not find the expansion.*

Solution. i. The function h has poles at $z = 0, \pm 1$. The Taylor series converges on the largest disk centered at $2+i$ on which h is analytic. The radius is the distance to the nearest singularity. The nearest singularity is 1.

$$\text{radius of convergence} \; = \; |(2+i)-1| = |1+i| = \sqrt{2}\,.$$

ii. The geometric series yields

$$\frac{1}{1-z^2} \; = \; 1 + z^2 + (z^2)^2 + (z^2)^3 + \cdots\,.$$

Therefore

$$h(z) \; = \; \frac{1}{z} + z + z^3 + z^5 \; + \cdots\,.$$

iii. The function h is analytic on two annuli with center at 0. They are $0 < |z| < 1$ and $1 < |z| < \infty$. The second of these is the answer.

2. (2+4 points). Suppose that $f(x)$ is a continuous complex valued function on $[0,2]$ with $|f(x)| \le 1$ for all $x \in [0,2]$. Define $g(z)$ as the Fourier integral

$$g(z) \; := \; \int_0^2 f(x)\, e^{izx}\, dx\,.$$

i. Show that g satisfies the exponential growth estimate

$$\big|g(z)\big| \;\le\; 2\,e^{2|\mathrm{Im}\,z|}\,.$$

ii. Show that $g(z)$ is an entire analytic funciton of z. State clearly any results that you are using.

Solution. i. Estimate

$$\left|\int_0^2 f(x)\,e^{izx}\,dx\right| \le \int_0^2 \left|f(x)\,e^{izx}\right|\,dx \le \int_0^2 |f(x)|\;\left|\,e^{i(u+iv)x}\right|\,dx\,, \qquad z=u+iv\,.$$

At this point one has to avoid the error of writing $z = x + iy$. The letter x already means something else. With this substitution the integral depends only on y so cannot be analytic.

Having avoided that trap, estimate

$$|f(x)|\le 1\,, \qquad \left|e^{iux}e^{-vx}\right| \le |e^{iux}|e^{-vx} = e^{-vx}\,.$$

Since $|x| \le 2$ on the interval, one has $e^{-vx} \le e^{2|v|}$ the bound being sharp when $x = 2$ and $v < 0$. This yields

$$|g(z)| \le e^{2|Imz|}\int_0^2 dx = 2\,e^{2|Imz|}\,.$$

ii. The integrand is a function $G(x,u,v)$ that is continuous and continuously differentiable with respect to u,v. Leibniz' rule implies that $g(u+iv)$ is has continuous partial derivatives with respect to u,v and

$$\frac{\partial g}{\partial u} \;=\; \int_0^2 \frac{\partial}{\partial u}\Big(f(x)\,e^{i(u+iv)x}\Big)\,dx \;=\; \int_0^2 ix\,f(x)\,e^{i(u+iv)x}\,dx\,,$$

$$\frac{\partial g}{\partial v} \;=\; \int_0^2 \frac{\partial}{\partial v}\Big(f(x)\,e^{i(u+iv)x}\Big)\,dx \;=\; \int_0^2 (-x)\,f(x)\,e^{i(u+iv)x}\,dx\,.$$

Therefore $ig_u = g_v$ verifying the Cauchy-Riemann equations. As the partials are continuous, the Cauchy-Riemann equations imply analyticity.

3. (4 points). **i.** Define

$$f(z) \;:=\; z(z-1)(z-2)\cdots(z-8)\,.$$

For integers $0 \le k$ define Γ_k to be the image by f of the circle $|z| = k + 0.5$ traversed in the postive (counter clockwise) sense. For $k = 0$, $k = 3$, and $k = 6$ determine how many times Γ_k winds about the origin in the positive sense?

Solution. The function f is everywhere analytic and has zeros exactly at the points $0, 1, 2, \ldots, 8$. The zeros are all simple.

The circles $|z| = k + 0.5$ with integer k pass through no zeros.

The argument principal implies that the number of times Γ_k winds around the origin in the positive sense is equal to the number of zeros lying inside $|z| = k + 0.5$ counted according to their multiplicity.

For $k = 0, 3, 6$ the circle incloses 1,4, and 7 zeros respectively so the winding numbers of Γ_0, Γ_3, and Γ_6 are 1,4, and 7 respectively.

4. (8 points). Evaluate exactly the Fourier integral

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1+x^2)^2}\, dx\,, \qquad \xi\ >\ 0\,.$$

You may leave the answer in terms of an easily computed derivative without computing the derivative. Make sure the point where the derivative needs to be evaluated is clear.

Solution. Defne $f(z) = e^{iz\xi}/(1+z^2)^2$. The function f is analytic in $\mathbb{C} \setminus \pm i$ and has a double pole at i and at $-i$. With $z = x + iy$

$$e^{iz\xi} \ =\ e^{i(x+iy)\xi} \ =\ e^{ix\xi} e^{-y\xi}\,.$$

Since $\xi \geq 0$ the exponential factor is bounded in the upper half space $\operatorname{Im} z = y \geq 0$.

For $R > 1$ denote by Ω_R the half disk in the upper half space centered at 0, and with radius R. f has a pole at i inside and is analytic at other points including the boundary. Write

$$\frac{e^{iz\xi}}{(z^2+1)^2} = \frac{e^{iz\xi}}{[(z+i)(z-i)]^2} = \frac{e^{iz\xi}}{(z+i)^2}\,\frac{1}{(z-i)^2}\,.$$

Cauchy's integral formula for derivatives implies

$$\oint_{\partial\Omega} f(z)\, dz = \oint_{\partial\Omega} \frac{e^{iz\xi}}{(z+i)^2}\,\frac{1}{(z-i)^2}\, dz \ =\ 2\pi i \left(\frac{d}{dz}\, \frac{e^{iz\xi}}{(z+i)^2} \right)_{z=i}. \tag{4.1}$$

Most use the residue theorem to find

$$\oint_{\partial\Omega_R} f(z)\, dz \ =\ 2\pi i\, \operatorname{Res}(f, i)\,. \tag{4.2}$$

To compute the residue, expand in a Taylor series centered at $\underline{z} = i$

$$\frac{e^{iz\xi}}{(z+i)^2} = A_0 + A_1(z-i) + A_2(z-i)^2 + \cdots\,, \tag{4.3}$$

$$f(z) = \frac{A_0}{(z-i)^2} + \frac{A_1}{z-i} + A_2 + \cdots, \quad \text{so} \quad \mathrm{Res}(f,i) = A_1\,. \tag{4.4}$$

Equations (4.2)-(4.4) reprove (4.1).

On the semicircular arc

$$|1+z^2| \geq |z|^2 - 1 = R^2 - 1, \qquad |f| \leq \frac{1}{(R^2-1)^2}\,.$$

Therefore

$$\left| \int_{semicircle} f(z)\, dz \right| \leq |\text{semicircle}| \ \frac{1}{(R^2-1)^2} = \frac{\pi R}{(R^2-1)^2} \ \to \ 0\,,$$

as $R \to \infty$. Therefore,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \ = \ \lim_{R\to\infty} \oint_{\partial\Omega_R} f(z)\ dz. \tag{4.2}$$

Since $|f(x)| \leq c/|x|^4$ for $|x| \geq 1$ and is continuous for $|x| \leq 1$ it follows that f is absolutely integrable so

$$\int_{-\infty}^{\infty} f(x)\, dx \ = \ \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \ = \ \left(\frac{d}{dz} \frac{e^{iz\xi}}{(z+i)^2} \right)_{z=i}.$$

The next computation was not required. The derivative inside parentheses is equal to

$$i\xi e^{iz\xi} \frac{1}{(z+i)^2} \ + \ e^{iz\xi} \frac{-2}{(z+i)^3}\,.$$

Evaluating at $z = i$ yields

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i e^{-\xi} \left(\frac{i\xi}{(2i)^2} - \frac{2}{(2i)^3} \right) = \pi e^{-\xi} \left(\frac{\xi}{2} + \frac{1}{2} \right).$$

A check on accuracy is that the answer must be positive for $\xi = 0$.

5. (7 points). Denote by Q the open quadrant $\big\{x > 0\,,\ y > 0\big\}$. Denote by $\overline{Q} := \big\{x \geq 0\,,\ y \geq 0\big\}$ the closed quadrant. Find the the unique bounded harmonic function $u(x,y)$ on Q that is continuous on $\overline{Q} \setminus \{i, 0\}$ and assumes the boundary values

$$\begin{aligned} u = 0 \quad &\text{when} \quad y = 0 \quad \text{and} \quad x > 0\,, \\ u = 1 \quad &\text{when} \quad x = 0 \quad \text{and} \quad 0 < y < 1\,, \\ u = 3 \quad &\text{when} \quad x = 0 \quad \text{and} \quad 1 < y < \infty\,. \end{aligned}$$

Solution. The function $w = z^2$ maps Q conformally to the upper half space with the separation points i and 0 on the boundary of Q mapped respectively to -1 and 0. The problem is solved by $u := v(z^2)$ provided that $v(x,y)$ is a bounded harmonic function in the upper half space satisfying the boundary conditions

$$\begin{aligned} v(x,0) &= 0 \quad \text{when} \quad x > 0\,, \\ v(x,0) &= 1 \quad \text{when} \quad 0 < x < 1\,, \\ v(x,0) &= 3 \quad \text{when} \quad x < -1\,. \end{aligned}$$

Seek $v(w)$ in the form

$$v \;=\; A_0 \;+\; A_1 \arg(w) \;+\; A_2 \arg\big(w - (-1)\big)\,, \qquad A_j \text{ constant}\,.$$

Evaluating for $x > 0$ on the x-axis and using the first boundary condition yields

$$0 = v(x,0) = A_0\,.$$

Evaluating for $-1 < x < 0$ using the boundary condition and the value of A_0 just determined yields

$$1 = v(x,0) = \pi A_1\,, \qquad A_1 = 1/\pi\,.$$

Finally for $x < -1$

$$3 = \pi A_1 + \pi A_2 = 1 + \pi A_2\,, \qquad A_2 = 2/\pi\,.$$

The solution is then

$$u \;=\; \frac{1}{\pi}\arg(z^2) \;+\; \frac{2}{\pi}\arg(z^2+1)\,.$$

Discussion. The solution u is bounded and continuous at all boundary points other than those where the boundary values are discontinuous. We proved that there is at most one such solution.

6. (8 points). Evaluate exactly

$$\int_0^\infty \frac{x^{1/5}}{1+x^2}\,dx\,.$$

You may leave the answer as a complicated algebraic expression in complex numbers.

Solution. The integral is absolutely convergent since the integrand is continuous and for $x \geq 1$ is $\leq c/x^{2-1/5}$.

Use the thermometer contour that has two arcs parallel to the x axis and $0 < \epsilon << 1$ units above and below. Plus a small circle of radius $r << 1$ almost encircling the origin, plus a large circle of radius $R >> 1$ almost encircling the origin. The contour bounds the domain $\Omega(r, R, \epsilon)$.

The function

$$f(z) \;=\; \frac{z^{1/5}}{1+z^2} \;=\; \frac{z^{1/5}}{(z-i)(z+i)} \tag{6.1}$$

is analytic in the domain inclosed except for poles at i and $-i$. The root is defined using the branch of arg so that $0 < \arg < 2\pi$. This branch of $z^{1/5}$ is defined on the plane slit by removing the non negative real axis. The residue theorem implies that

$$\oint_{\partial\Omega} f(z)\; dz \;=\; 2\pi i\Big[\mathrm{Res}(f,i) \;+\; \mathrm{Res}(f,-i)\Big]\,.$$

For ϵ small the integrals over the horizonal segment above the x axis is nearly equal to

$$\int_r^R \frac{x^{1/5}}{1+x^2}\; dx$$

while the integral over the other horizontal segment is nearly equal to

$$-\,e^{2\pi i/5}\int_r^R \frac{x^{1/5}}{1+x^2}\; dx\,.$$

The minus sign is because the curve is traversed the opposite sense. The exponential is because the fifth root is $e^{2\pi i/5}x^{1/5}$. Thus passing to the limit $\epsilon \to 0$ shows that

$$-\oint_{|z|=r} dz + \oint_{|z|=R} dz + (1-e^{2\pi i/5})\int_r^R \frac{x^{1/5}}{1+x^2}\; dx \;=\; 2\pi i\Big[\mathrm{Res}(f,i) \;+\; \mathrm{Res}(f,-i)\Big]\,.$$

The integrand is bounded for $|z| \le 1$ so the integral over $|z| = r$ is no larger that $2\pi r$ times that bound so tends to zero as $r \to 0$.

For $|z| = R$ one has

$$|z^{1/5}| = R^{1/5}\,, \qquad \left|\frac{1}{1+z^2}\right| \;\le\; \frac{1}{R^2-1}\,.$$

Therefore

$$\left|\oint_{|z|=R} dz\right| \;\le\; \frac{2\pi R\; R^{1/5}}{R^2-1} \;\to\; 0$$

as $R \to \infty$. Therefore sending $r \to 0$ and $R \to \infty$ using the absolute integrability yields

$$(1-e^{2\pi i/5})\int_0^\infty \frac{x^{1/5}}{1+x^2}\; dx \;=\; 2\pi i\Big[\mathrm{Res}(f,i) \;+\; \mathrm{Res}(f,-i)\Big]\,. \tag{6.2}$$

Since the poles are simple, formula (6.1) yields

$$\mathrm{Res}(f,-i) = \left(\frac{z^{1/5}}{z-i}\right)_{z=-i} = \frac{e^{(3/2)i\pi/5}}{-2i}\,, \qquad \mathrm{Res}(f,i) = \left(\frac{z^{1/5}}{z+i}\right)_{z=i} = \frac{e^{(\pi/2)i/5}}{2i}\,.$$

Inserting this in (6.2) yields

$$\int_0^\infty \frac{x^{1/5}}{1+x^2}\,dx \;=\; 2\pi i\left[\frac{e^{i\pi/10}}{2i} \;-\; \frac{e^{3i\pi/10}}{2i}\right](1-e^{2\pi i/5})^{-1}\,.$$

This is the complicated but acceptable expression. Simplifying yields,

$$=\;\pi\left[\frac{e^{i\pi/10}-e^{3i\pi/10}}{1-e^{2\pi i/5}}\right] \;=\; \pi\,\frac{e^{2i\pi/10}}{e^{2i\pi/10}}\,\frac{e^{-i\pi/10}-e^{i\pi/10}}{e^{-i2\pi/10}-e^{2\pi i/10}} \;=\; \pi\,\frac{\sin(\pi/10)}{\sin(\pi/5)}\,.$$

7. (2+2+2 points). **i.** The function $g(z) := e^{1/z}-1$ satisfies $g(z_n)=0$ for

$$\frac{1}{2\pi i n} \;:=\; z_n \;\to\; 0\,.$$

Nevertheless, g is not identically equal to zero. Explain why this does not contradict the unique continuation prinicipal for analytic functions.

ii. Does $1/\sqrt{z}$ have a pole at $z=0$? Explain.

iii. Compute for integer $n\ge 1$,

$$\oint_{|z|=1} \overline{z}^n\;dz\,,$$

where the contour is traversed in the positive sense.

Solution. **i.** The unique continuation principal asserts that the zeros of an analytic function cannot converge to a point of analyticity. These zeros converge to 0 and g is not analytic on any disk centered at 0.

ii. No because $z=0$ is not an isolated singularity of $1/\sqrt{z}$. The square root and one over the square root cannot be defined as a continuous function an any punctured disk centered at the origin.

iii. Parametrize by $z(\theta)=e^{i\theta}$, $0<\theta<2\pi$. Then

$$\oint_{|z|=1} \overline{z}^n\;dz = \int_0^{2\pi} \left(\overline{e^{i\theta}}\right)^n\,\frac{dz}{d\theta}\;d\theta = \int_0^{2\pi} e^{-in\theta}\;i\,e^{i\theta}\;d\theta$$

The integral is equal to $2\pi i$ for $n=1$ and is equal to 0 otherwise.