Math 555, Fall 2011 Prof. J. Rauch

Midterm Exam October 20, 2011

Instructions. 1. Closed book. Two sides of a 3.5in. $\times$ 5in. sheet of notes from home.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are 7 questions. They consist of 14 short subquestions each worth 5 points.
70 points total. You have about 5.5 minutes per short question. Be efficient.

1. (5+5 points). **i.** For the function $f(z) = 1/z$ compute $f(1+i)$ and $f'(1+i)$.

ii. Find the local magnification factor and rotation angle of f at $z = 1+i$.

Solution. i.

$$f(1+i) \;=\; \frac{1}{1+i} \;=\; \frac{1}{1+i}\frac{1-i}{1-i} \;=\; \frac{1-i}{2} \;=\; \frac{1}{\sqrt{2}}\, e^{-\pi/4}\,.$$

$$f'(z) = \frac{-1}{z^2}\,, \quad z = 1+i = \sqrt{2}\, e^{\pi/4}\,, \quad \frac{1}{z} = \frac{1}{\sqrt{2}}\, e^{-\pi/4}\,, \quad f'(1+i) = -1/2e^{-\pi/2} = i/2 = \frac{1}{2}e^{i\pi/2}\,.$$

ii.

$$\text{Magnification factor} \;:=\; |f'(1+i)| \;=\; \frac{1}{2}\,.$$

$$\text{Rotation factor} \;:=\; \arg f'(1+i)| \;=\; \frac{\pi}{2}\,.$$

Discussion. i. This question tests computational skill with complex numbers. **ii.** This tests the geometric interpretation of the complex derivative.

2. (5+5 points). **i.** Let Γ denote the contour connecting $(0,0)$ to $(1,1)$ along the straight line $y = x$. Use the definition of line integral evaluate

$$\int_\Gamma |z|^2 \, dz \, .$$

ii. If $f(z)$ is an analytic function on an open connected set Ω. State clearly what "independence of path" has to do with antiderivatives of analytic functions on Ω.

Solution. i. Parameterize the curve. The simplest is $z(t) = t + it$ for $0 \le t \le 1$. Then

$$|z(t)|^2 \;=\; t^2 + t^2 \;=\; 2t^2 \,, \qquad \frac{dz}{dt} \;=\; 1 + i \, .$$

The definition of line integrals dz yields

$$\int_0^1 |z(t)|^2 \, \frac{dz(t)}{dt} \; dt \;=\; \int_0^1 2t^2 \; (1+i) \; dt \;=\; \frac{2(1+i)}{3} \, .$$

ii. A function F on Ω is an antiderivative of the analytic function f when $F' = f$. If f has an antiderivative and C is an arc in Ω with starting point P and endpoint Q then the fundamental theorem of calculus asserts that

$$\int_C f(z) \; dz \;=\; F(Q) - f(P)$$

so is independent of the path connecting P to Q. Thus independence of path is a necessary condition for the existence of an antiderivative. This necessary sufficient is also sufficient.

Alternative shorter version. An analytic function f on Ω has an antiderivative in Ω if and only if for arcs C in Ω, $\int_C f(z) \; dz$ depends only on the endpoints of C and is independent of the path joining the endpoints.

3. (5+5 points). **i.** Consider the function $f(z) = (z-1)^3$. At what point(s) z of the annulus $1 \le |z| \le 2$ does the function $|f(z)|$ attain its maximum? Find the maximum.

ii. If $g(z)$ is analytic in the annulus and $|g(z)| \le 10$ for z in the annulus, give an upper bound on the possible values of for $g'(3/2)$.

Solution. i.

$$|f(z)| \;=\; |z-1|^3 \;=\; \text{dist}(z,1)^3$$

so the maximum is attained at the points of the annulus that are farthest from the point 1. There is a unique such point, $z = -2$. For this point,

$$|f(-2)| \;=\; 3^3 \;=\; 27\,.$$

The maximum value is 27.

Alternate more careful version. The triangle inequality implies that $|z-1| \le |z| + 1$ so

$$|z-1|^3 \le (|z|+1)^3$$

with equality if and only if z is a nonpositive real number. In our domain, $|z| \le 2$ so 27 is an upper bound. The bound is achieved at the unique negative real of length 2 that belongs to the annulus.

ii. Cauchy's inequalities imply that if $|g(z)| \le M$ for all $z \in G$ then

$$|g^n(z)| \;\le\; \frac{n!\,M}{\text{dist}\,(z,\partial G)^n}\,.$$

Use this estimate for $z = 3/2$ with distance $1/2$ to the boundary of the annulus, $M = 10$, and $n = 1$ to find

$$|g'(3/2)| \;\le\; \frac{10}{1/2} \;=\; 20\,.$$

4. (5+5 points). Let $\mathbb{D}$ denote the disk of radius 2 and center at the origin. Denote by $\partial\mathbb{D}$ its boundary.

i. Use a Cauchy integral theorem to evaluate

$$\oint_{\partial\mathbb{D}} (z-i)^n \ e^z \ dz$$

for integers $n \geq 0$.

ii. Use a second Cauchy integral theorem to evaluate for $n < 0$.

You can check your answers using the Taylor series of e^z, but these questions are about the integral theorems.

Solution. i. For $n \geq 0$ the function $(z-i)^n e^z$ is analytic throughout the disk $\mathbb{D}$ including the boundary. By Cauchy's theorem, the integral around the boundary of a domain on which a function is analytic vanishes.

$$n \geq 0 \quad \Longrightarrow \quad \oint_{\partial\mathbb{D}} (z-i)^n \ e^z \ dz \ = \ 0 \, .$$

ii. For $n < 0$ use Cauchy's integral formula for $k \geq 0$

$$\oint_{\partial R} \frac{f(z)}{(z-w)^{k+1}} \ dz \ = \ \frac{2\pi i f^{(k)}(w)}{k!} \, .$$

Take $f(z) = e^z$, $w = i$, and $k+1 = |n| = -n$ so the left hand side is the desired integral. Then $k = -n-1$ and $d^k e^z / dz^k = e^z$ so

$$n \ < \ 0 \quad \Longrightarrow \quad \oint_{\partial\mathbb{D}} (z-i)^n \ e^z \ dz \ = \ \frac{2\pi i}{k!} \left. \frac{d^k e^z}{dz^k} \right|_{z=i} \ = \ \frac{2\pi i}{(-n-1)!} \ e^i \, .$$

5. (5+5 points). **i.** Use the inverse function theorem to show that the function $f(z) = \sin z$ is invertible on a neighborhood of $z = 2i$.

ii. Find the largest open disk with center at $z = 0$ on which the Taylor series of $(\cos z)^{-1}$ converges.

Solution. i. The inverse function theorem asserts the local invertibility of an analytic function on a neighborhood of each point $\underline{z}$ so that $f'(\underline{z}) \neq 0$. Therefore it suffices to show that $f'(2i) \neq 0$.

Since $f'(z) = \cos z$ it suffices to show that $\cos 2i \neq 0$ We know that the roots of $\cos z = 0$ are exactly the points $z = \pi/2 + \pi n$ with $n \in \mathbb{Z}$. In particular $\cos 2i \neq 0$ since all the roots are on the real axis.

Alternatively compute

$$\cos 2i = \frac{e^{i(2i)} + e^{i(-2i)}}{2} = \frac{e^{-2} + e^{2}}{2}.$$

The numerator is the sum of two strictly positive reals therefore strictly postive. Therefore $e^{2i} > 0$.

ii. Since $\cos 0 \neq 0$, $g(z) := 1/\cos z$ is analytic on a neighborhood of the origin.

Taylor series of g converges on the largest open disk D with center at the origin on which g is analytic. That is the largest such disk that does not contain a zero of $\cos z$.

Since the zeros of $\cos z$ are the points $\pi/2 + \pi n$, the closest to the origin are $\pm\pi/2$. Therefore the radius of convergence of the Taylor series is equal to $\pi/2$.

6. (5+5 points). **i.** $k(z)$ is analytic on the complement of the pair of points 0 and 1. Explain why k has two Laurent series expansions in powers of z^n, that is series centered at $z = 0$.

ii. Consider

$$\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{k(z)}{z^3}\, dz\,.$$

This integral evaluates which coefficient in which Laurent expansion?

i. The function $k(z)$ is analytic in each of the two annulli

$$0 < |z| < 1\,, \qquad \text{and} \qquad 1 < |z| < \infty\,.$$

The second is degenerate in that the second radius is infinite. $k(z)$ has two Laurent expansions, one convergent in the first annullus and the second in the second annulus.

ii. Consider the Laurent expansion

$$k(z) \;=\; \sum_{n=-\infty}^{n=\infty} c_n z^n\,, \qquad 0 < |z| < 1\,.$$

uniformly convergent on compact subsets of $0 < |z| < 1$. In particular $k(z)/z^3$ has the uniformly convergent expression

$$\sum_n c_n\, z^{n-3}$$

on $|z| = 1/2$.

Integrating yields

$$\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{k(z)}{z^3}\, dz \;=\; \frac{1}{2\pi i}\oint_{|z|=1/2} \sum_n c_n\, z^{n-3}\, dz\,.$$

The uniform convergence justifies interchanging sum and integral to obtain

$$\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{k(z)}{z^3}\, dz \;=\; \frac{1}{2\pi i}\sum_n \oint_{|z|=1/2} c_n\, z^{n-3}\, dz\,.$$

The integrals on the right all vanish except that with $n = 2$ yielding

$$\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{k(z)}{z^3}\, dz \;=\; c_2\,.$$

7. (5+5 points). **i.** Find the order of the pole of

$$f(z) = \frac{\sin z}{(1-\cos z)^2}$$

at $z = 0$.

ii. If the pole has order m, find the coefficient of z^{-m} in the Laurent expansion of f.

Solution. i. Expand numerator and denominator in Taylor series centered at $z = 0$,

$$\sin z \;=\; z \;-\; \frac{z^3}{3!} \;+\; \frac{z^5}{5!} \;\cdots$$

$$1-\cos z \;=\; 1 \;-\; \Big(1 \;-\; \frac{z^2}{2!} \;+\; \frac{z^4}{4!}\;\cdots\Big) \;=\; \frac{z^2}{2!} \;-\; \frac{z^4}{4!}\;\cdots$$

Factor the largest powers of z,

$$\sin z \;=\; z\Big(1 \;-\; \frac{z^2}{3!} \;+\; \frac{z^4}{5!}\;\cdots\Big)$$

$$1-\cos z \;=\; z^2\Big(\frac{1}{2!} \;-\; \frac{z^2}{4!}\;\cdots\Big),$$

$$(1-\cos z)^2 \;=\; z^4\Big(\frac{1}{4} \;-\; \frac{z^2}{4!}\;\cdots\Big),$$

Therefore

$$\frac{\sin z}{(1-\cos z)^2} \;=\; \frac{1}{z^3}\,\frac{1 \;-\; \frac{z^2}{3!} \;+\; \frac{z^4}{5!}\;\cdots}{\frac{1}{4} \;-\; \frac{z^2}{4!}\;\cdots} \;:=\; \frac{1}{z^3}\,h(z)\,.$$

Since $h(z)$ is the quotient of two analytic functions, neither vanishing at the origin it follows that $h(z)$ is analytic in a neighborhood of 0. From the definition $h(0) = 4$. Therefore

$$\frac{\sin z}{(1-\cos z)^2} \;=\; \frac{1}{z^3}\,\Big(4 \;+\; c_1 z \;+\; c_2 z^2 \;+\; \cdots\Big).$$

The pole is of order 3.

ii. The coefficient of z^{-3} is 4.