Math 555, Fall 2011 Prof. J. Rauch

Homework 1. Due Thursday Sept 15

There are three important pieces of background to buttress at the start of a complex variables course; ideas from the differential calculus of maps from the plane to itself, complex numbers, and concepts of convergence of sequences and series of functions.

The first problems of the first two assignments remind you how partial derivatives can be used to analyse the behavior of functions mapping the plane to itself. They have a computational and geometric side. The matrix of partial derivatives

$$J \;=\; \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

called the Jacobian matrix after Jacobi. For points x, y near $\underline{x}, \underline{y}$ one has

$$(\Delta u, \Delta v) \;\approx\; J(\underline{x}, \underline{y})\,(\Delta x, \Delta y)\,.$$

This fundamental relation is the two dimensional analogue of $\Delta y \approx f'(\underline{x})\Delta x$ that is the key relation in differential calculus of functions of one variable.

1. i. For inversion in the unit circle defined by

$$0 \;\neq\; (x, y) \quad\to\quad (u, v) \;:=\; \left(\frac{x}{x^2+y^2}\,,\,\frac{y}{x^2+y^2}\right)$$

compute the Jacobian matrix.

ii. For the mapping

$$(x, y) \;\mapsto\; (x - y\,,\, x^2 + y^2),$$

compute the Jacobian matrix.

iii. Verify that for all (x, y) the Jacobian of the mapping in **i** is a matrix that maps circles to circles. Verify that for most (x, y) the Jacobian of the mapping in **ii** does not have this property.

Discussion. For **i** infinitesimal circles are sent to infinitesimal circles. That is a circle of radius $r << 1$ centered at a point $\underline{x}, \underline{y}$ is sent to a curve that is very close (error $\sim r^2$) to a circle centered at the image of $(\underline{x}, \underline{y})$. The images that might have been ellipses with any eccentricity, have eccentricity is equal to 1. For **ii**, the ellipses are nearly always noncircular.

2. 11/2a (This means problems 2a on page 11 of the text.)

3. 12/15b,d. **Hints.** Use polar form. If you know one summand for part d, then you know the other.

4. 12/20g.

5. 21/6. **Hint.** Use continuity of $0 \le x \mapsto \sqrt{x}$.

6. 21/12b.

7. 92/11 (variant). Find a sequence of continuous functions on $[0, 1]$ that converges but not uniformly, and so that the limit is continuous. This complements the example in equations (19), (20) on page 84.