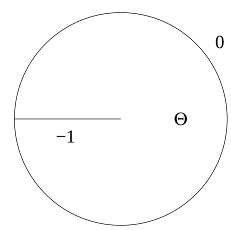
Math 555, Fall 2011 **J. Rauch**
Homework 12 **Due December 8**

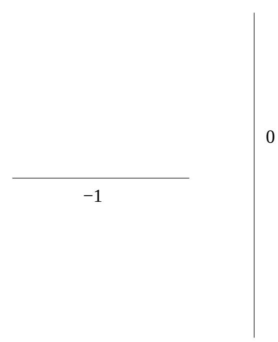
1. 243/3.

2. 243/4.

3. 245/19. The answer given is the unique bounded solution. There are many others. Find the bounded solution.

4. 245/18 Variant. *Denote by* Γ *the plane* $\mathbb{R}^2$ *with the segments* $-\infty < x \le -\sigma$ *and* $\sigma \le x < \infty$ *on the* x*-axis removed as in the figure. Find the unique temperature* $u(x,y)$ *harmonic and bounded on* Γ *with* $u = -1$ *on the left hand removed half line and* $u = 1$ *on the right hand removed half line.* **Hint.** We know how to find a bounded harmonic function in the upper half disk that vanishes on the diameter and is equal to one on the top of the disk.





Consider the domain Θ equal to the unit disk with center at the origin with the interval $-1 < x \le 0$ on the x-axis removed. The map $z^{1/2}$ suitably defined maps Θ to the right half disk. This allows one to solve the Dirichlet problem on the slit disk Θ with values 1 on both sides of the slit and value 0 on the circumference of Θ. Send the intersection of the slit with the circumference to infinity to solve a heat flow problem sketched in the third figure. Finish by using symmetry of the original problem with respect to the y-axis.

The next three problems concern irrotational, incompressible, inviscid steady planar fluid flows. We call them simply flows.

The next two problems concern flows in the wedge $0 < \theta < A$ where $A < \pi/4$ and the flow is required to be tangent to the bounding half lines. We find the unique complex potentials $F(z)$ defining such flows and having the desirable dilation scaling that for any $\sigma > 0$,

$$F(\sigma z) \;=\; c(\sigma)\, F\,, \tag{1}$$

for a suitable real $c(\sigma)$ depending on σ.

5. i. Show that if F is analytic in the wedge and satisfies (1), then if σ and τ are two positive constants then $c(\sigma\tau) = c(\sigma)c(\tau)$. **ii.** Show that c is a differentiable function on $]0,\infty[$. **Hint.** Consider a single fixed z. **iii.** Show that F is a homogeneous function of z. **Hint.** Use logarithms to nearly determine $c(\sigma)$. **iv.** Show that an analytic F on the wedge is of the form $b\,z^\alpha$ for some real α and complex b. **Hint.** Read the earlier homework problem identifying analytic functions homogeneous of degree n with n integer.

6. Find all flows in the wedge whose flow is parallel to the bounding lines and satisfies the symmetry (1). **Hint.** Use the result of the preceding problem. **Discussion.** The flow velocity is bounded at the corner but the derivatives of the velocity are unbouded at the corner. The case of $A = \pi/4$ does not have this divergence.

7. i. Starting with flow in the unit disk swirling about the origin, find a swirling flow in the upper half disk so that the flow swirls about the point midway between the circle center and the circumference. **ii.** Show that near the corners of the half disk the flow resembles the flow with complex potential z^2 in a quadrant.