Math 555, Fall 2011 **J. Rauch**
Homework 13 **Never due**

1. For each of the following functions determine if the point at infinity on the Riemann sphere is removeable, essential, or a pole. The function $p(z)$ designates a polynomial.

a. $\sin z$, **b**. $\sin(1/z)$, **c**. $p(z)\sin z$, **d**. $p(1/z)\,\sin z$, **e**. $p(z)\,\sin 1/z$, **f**. $p(1/z)\sin 1/z$.

2. Explain why the prinicipal branch of $\ln z$ and $\sqrt{z}$ do not have an isolated singularity at the point at infinity of the Riemann sphere.

3. Show that the function $(x_2 + ix_3)^2$ is not an analytic function on the Riemann sphere.

4. 245/15.

5. 212/3.

6. 212/8. That is, **i.** construct a bounded harmonic function in the exterior of the disk with arbitrary smooth boundary values on the boundary of the disk. **ii.** Prove uniqueness by expanding

$$u \;=\; \sum_n a_n(r) e^{in\theta}$$

in a Fourier series in r. Show that for integer $n > 0$ the function

$$a_n(r) \;:=\; \int_0^{2\pi} f(r\,e^{i\theta}) e^{-in\theta}\; d\theta$$

satisfies $r(r\,a_n')' - n^2 a_n \;=\; 0$ and that the only bounded solution of this equation that vanishes for $r = 1$ is $a_n = 0$.

7. 212/9 variant. If h is a smooth function on $\{|z| = 1\}$ with $\int_0^{2\pi} h(e^{i\theta})\, d\theta = 0$ then there is exactly one solution of the Neumann problem on the disk that is

$$\frac{\partial u}{\partial r}\bigg|_{r=1} \;=\; h$$

with

$$\int_0^{2\pi} u(r\,e^{i\theta})\; d\theta \;=\; 0.$$

9. 213/21. **Hint.** Use reflection to extend the map to a map from punctured disk to punctured disk. This takes an infinity of reflections. Then show that the singularity at the origin is removable.