Math 555, Fall 2011 Prof. J. Rauch

Homework 2. Due Thursday September 22

1. 44/10

2. 44/11b,d.

3. 44/14/b,c.

Big Picture. The next problems reinforce the fact that a continuously differentiable function $\mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ is differentiable in the complex sense if and only if the Cauchy-Riemann equations $i\,\partial f/\partial x = \partial f/\partial y$ are satisfied.

4. Prove that if $f(z)$ is analytic on an open connected set Ω and $\overline{f(z)}$ is also analytic, then f is constant. **Remarks.** 1. In particular the only real valued analytic functions are constants. 2. There is a close cousin assertion that that if f is analytic on Ω and $f(\overline{z})$ is analytic on the complex conjugate of Ω then f is constant. You are not asked to show this in this problem. These results correspond to 44/13.

The next two problems prove a result that was stated in class. If you have trouble with any part you should proceed to the next assuming the result of troublesome part(s).

5. Denote by Ω either a nonempty angular sector $\Omega := \big\{z \neq 0 \;:\; a < arg(z) < b\big\}$ where $0 < b - a < 2\pi$ and arg is a branch of the argument, or, $\Omega = \mathbb{C} \setminus 0$.

Definition $f : \Omega \to \mathbb{C}$ *is positive homogeneous of degree* $n \in \mathbb{R}$, *if for all* $\sigma > 0$ *and* $z \in \Omega$,

$$f(\sigma\, z) \;=\; \sigma^n\, f(z)\,. \tag{1}$$

Show that if f is an analytic function on Ω that is positive homogeneous of degree n then for all $z \in \Omega$,

$$z\, f'(z) \;=\; n\, f(z)\,.$$

Hint. Differentiate (1) with respect to σ. Then set $\sigma = 1$. Alternatively, evaluate the derivative of f by computing

$$\lim_{h \to 0^+} \frac{f(z(1+h)) - f(z)}{hz}\,, \qquad (n.b.\;\; hz = \Delta z)$$

one way using the definition of derivative and a second way using homogeneity. **Discussion.** The alternative approaches are really the same.

6. i. Continuing show that if $f : \Omega \to \mathbb{C}$ is positive homogeneous of degree n and analytic then there is a complex constant c so that $f = c\, z^n$ for all $z \in \Omega$. **Hint.** Show that $z^{-n} f$ is constant by differentiation.

ii. Show that if

$$P(x,y) \;=\; \sum_{j=0}^{n} a_j\, x^j\, y^{n-j}\,, \qquad a_j \;\in\; \mathbb{C}$$

is a complex polynomial containing only monomials of degree, n then P is analytic if and only if $P = c\, z^n$ for some c. **Hint.** For the only if direction use the result in **i.** There is a problem **7** on the next page.

7. 73/10. Also evaluate the line integrals

$$\int_C |z|\, dx, \qquad \text{and,} \qquad \int_C |z|\, ds\,,$$

along the same curves.