Math 555 Fall 2011
Homework 3
Due September 29

1. i. Suppose that $\Omega \subset \mathbb{C}$ is open, $f = u + iv : \Omega \to \mathbb{C}$ is analytic, $\underline{z} = \underline{x} + i\underline{y} \in \Omega$, and, $f'(\underline{z}) \neq 0$. Show that both $\operatorname{grad} u(\underline{x}, \underline{y}) \neq 0$ and $\operatorname{grad} v(\underline{x}, \underline{y}) \neq 0$.

The Implicit Function Theorem then implies that each of the level curves

$$\big\{(x,y) \ : \ u(x,y) = u(\underline{x}, \underline{y})\big\} \qquad \text{and,} \qquad \big\{(x,y) \ : \ v(x,y) = v(\underline{x}, \underline{y})\big\}$$

are smooth curves near $(\underline{x}, \underline{y})$. For example, if $u_y(\underline{x}, \underline{y}) \neq 0$ then the IFT implies that the level set of u is locally a smooth graph $y = h(x)$ with h infinitely differentiable. (Here we use the fact that u and v are infinitely differentiable.)

ii. Show that the level curves $u = u(\underline{x}, \underline{y})$ and $v = v(\underline{x}, \underline{y})$ are orthogonal at $\underline{z}$. **Hint.** Show that the normal vectors to the level curves are orthogonal. **Discussion.** If you graph the system of level curves of u and those of v you have curves that intersect at right angles. They define an orthogonal local coordinate system.

2. Show that $e^{\overline{z}}$ is not analytic. **Hint.** $e^x e^{-iy}$.

4. i. Show that if $m \neq n$ are integers then

$$\int_0^{2\pi} e^{im\theta} \ e^{-in\theta} \ d\theta \ = \ 0. \tag{1}$$

ii. With the same m, n evaluate

$$\int_C z^m \ \overline{z}^n \ dz$$

where C is the unit circle traversed in the positive sense.

Discussion. Identity (1) is important for Fourier series. It asserts that the functions $e^{im\theta}$ and $e^{in\theta}$ are orthogonal in the L^2 scalar product on 2π periodic functions. The scalar product is defined as

$$(g, h) \ = \ \int_0^{2\pi} g(\theta) \ \overline{h(\theta} \ d\theta \, .$$

5. 73/9.

6. 74/15.

7. i. Suppose that $f : \{0 < |z| < 1\} \to \mathbb{C}$ is analytic on the punctured unit disk. If $C = \{|z - z_0| = \rho\}$ is a circle in the punctured disc turning once about the origin in the positive sense, show that $\oint_C f(z) \ dz$ is equal to the integral of $f \ dz$ about the postively oriented circle $|z| = r < 1$ where r is so large that C lies entirely inside $|z| = r$. **ii.** Same question for C a rectangle turning once about the origin. **Hint.** Cauchy's Theorem.