Math 555 Fall 2011 Prof. J. Rauch
Homework 6 **Due Tuesday October 25**

There is a midterm exam on Thursday 20 October and this homework is NOT due then. We switch to Tuesday due dates to avoid a simultaneous exam and homework. It also solves the problem of Thanksgiving that falls on Thursday.

Reminder. Partial fraction is testable on the midterm. Fourier series is not.

Hint. Use stealth and cunning to compute series. Resort to general formulas for the coefficients is a method of last resort. Always try to derive new series from the basic ones for example $1/(1-z)$ and e^z by differentiation, integration, substitution, division, ... *etc.*

1. Insert $z = re^{i\theta}$ in $(1-z)^{-1} = \sum z^n$ to derive

$$\sum_{n\geq 1} r^n \cos n\theta \;=\; \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}\,.$$

2. Derive the Taylor expansion of $f(z) = 1/(1-z)$ about the point $\underline{z} = i$. **Hint.** Write $1 - z = (1-i) - (z-i)$. Factor the larger summand $1 - i$.

3. Find the Laurent expansion of $1/(z-1)(z-2)$ valid in $|z| > 2$.

4. i. The function

$$f(z) \;=\; \frac{1}{z^2(1-z)}$$

has four Laurent expansions. Sketch the regions where each of the four expansions are valid.

ii. Compute the partial fraction decomposition of f by computing two of Laurent expansions. Specify the regions in which the expansions are valid.

5. The Taylor expansion

$$\frac{1}{w} \;=\; \sum_{n=0}^{\infty} (-1)^n\,(w-1)^n\,, \qquad |w-1| < 1\,,$$

is derived by substitution as in Problem 3. Since the disk is simply connected the function has a unique antiderivative F defined on in $|w-1| < 1$ and satisfying $F(1) = 0$.

i. Integrate the uniformly convergent series over any contour in $|w-1| < 1$ connecting $w = 1$ to w find the Taylor series of F.

ii. From $F'(w) = 1/w$ with $F(1) = 0$ identify the function F.

6. Use long division to show that the Laurent series of $1/\sin z$ valid in $0 < |z| < 1$ begins

$$\frac{1}{\sin z} \;=\; \frac{1}{z} \;+\; \frac{z}{3!} \;+\; \Big[\frac{1}{(3!)^2} \;-\; \frac{1}{5!}\Big] z^3 \;+\; \cdots\,.$$

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