Math 555, Fall 2011 **J. Rauch**
Homework 9 **Due November 15**

1. i. Show that the function $1/\sin z$ has simple poles at the points $n\pi$ with residue equal to $\cos n\pi = (-1)^n$.

ii. Show that $\cos \pi z/ \sin \pi z$ has poles at the integers n with residues equal to $1/\pi$.

iii. Let C_N be the circle of radius $N + 1/2$ with center equal to the origin. Show that

$$\lim_{N\to\infty} \oint_{C_N} \frac{1}{z^2} \frac{\cos \pi z}{\sin \pi z}\, dz \;=\; 0.$$

2. iv. Compute

$$\mathrm{Res}\left(\frac{1}{z^2} \frac{\cos \pi z}{\sin \pi z}\,,\, 0 \right).$$

v. Apply the Residue Theorem to compute $\sum_{n=1}^{\infty} 1/n^2$.

Discussion. This is an application of a broadly applicable method to sum infinite series using residues. It sums for example $\sum_{n=-\infty}^{\infty} P(n)/Q(n)$ for polynomials P, Q so that Q has no real roots and $\deg Q \geq \deg P + 2$.

3. Prove the following result.

Theorem. *If $p(z)$ is a polynomial of degree n,*

$$p(z) \;=\; a_n z^n \;+\; \cdots \;+\; a_1 z + a_0, \qquad a_n \neq 0,$$

then there is an $R > 0$ so that for all w with $|w| > R$ the equation

$$p(z) \;=\; w$$

has n **distinct** *roots.*

Hints. i. Use Rouché's Theorem with $f = a_n z^n$. **ii.** You must choose your own contour(s). **iii.** Show that for large w, $f(z) = w$ has n simple roots far from each other. Show that $p(z)$ has nearby simple roots by choosing appropriate disks centered at the roots of f and showing that p has one root in each. **iv.** If you really want to optimize, try to see how close the solutions are to the solutions of $f(z) = w$. It is not hard to show that the distance between the roots of f and p is $\leq C$ for all $|w| \geq R$. You do not need to prove this sharper result.

4. If $c \in \mathbb{C}$ with $|c| > e$ show that the equation $c\, z^n = e^z$ has n roots counting multiplicity inside the unit circle.

5. Let Ω_R denote the part of the disk of radius R and center 0 belonging to the angular sector $0 < \theta < 2\pi/3$. Using the contour $\partial\Omega_R$ evaluate the integral

$$\int_0^{\infty} \frac{1}{x^3 + 1}\, dx \;=\; \frac{2\pi}{3\sqrt{3}}\,.$$

Discussion. This shows that with cunning you can do some integrals that look like the residue method does not work.

6. 190/23a.

7. 190/23d.