Math 555, Fall 2013, Prof. J. Rauch NAME ..........................................

**Final Exam, December 19, 2013**

**Instructions. 1.** Closed book. Two 3.5in. × 5in. sheet (four sides) of notes from home.
**2.** No electronics, phones, cameras, . . . etc.
**3. Show work and explain clearly.**
**4.** There are 9 questions on 9 pages. 50 points total.

**1.** (5 points). Suppose that $\mathcal{R} \subset \mathbb{C}$ is a nice bounded domain with boundary oriented in the standard way. Show that

$$\oint_{\partial\mathcal{R}} x \ dz \ = \ i \, \text{Area}\,(\mathcal{R})\,.$$

**Hint.** This tests the derivation of Cauchy's Theorem.

**Solution.** Green's Theorem that asserts that

$$\oint_{\partial\mathcal{R}} P(x,y)\,dx + Q(x,y)\,dy \ = \ \int\!\!\int_{\mathcal{R}} \frac{\partial Q}{\partial x} \ - \ \frac{\partial P}{\partial y}\,dx\,dy\,.$$

Write $xdz = x\,(dx + idy)$ and apply the formula with $P = x, \ Q = ix$ to find

$$\oint_{\partial\mathcal{R}} x \ dz \ = \ \int\!\!\int_{\mathcal{R}} i\,dxdy \ = \ i\,\text{Area}\,(\mathcal{R})\,.$$

**2.** (4 points). Is there a nonempty open disk $D$ on which the function

$$\Big(\frac{x^2}{2} \ + \ xy\Big) \ + \ i\,\Big(xy \ + \ \frac{y^2}{2}\Big)$$

is analytic?

**Solution.** The Cauchy-Riemann equations assert that $f_x + if_y = 0$ for analytic functions. Compute

$$\frac{\partial f}{\partial x} \ = \ (x+y) + iy\,, \qquad \frac{\partial f}{\partial y} \ = \ x + i(x+y)\,.$$

Therefore

$$\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y} \ = \ 0 + i(x+y)\,.$$

This vanishes only on the line $x + y = 0$ and therefore on no nonempty open disk.

**Discussion.** The Cauchy-Riemann equations at a point don't yield anything. Analyticity at a point **means** in a disk with center at the point.

**3.** (4+2 points). **i.** Define with integer $n \geq 1$,

$$g(z) \;:=\; z^n \;+\; \frac{1}{z-i}\,.$$

For $R > 2$, how many times does the image by $g(z)$ of the circle $\{|z| = R\}$ wind around the origin in the positive sense? Explain.

**ii.** How many roots $z$ of $g(z) = 0$ are there in the complex plane?

**Solution. i.** On $|z| = R > 1$ one has

$$|z^n| = R^n\,, \qquad |1/(z-i)| \;\leq\; 1/(R-1)\,.$$

For $R > 2$ the first is strictly larger than the second.

The dog on a leash principal implies that the image by $g$ winds that same number of times as the image by $z^n$. The image by $z^n$ winds $n$ times. **Ans.** $= n$.

**ii.** The argument principal asserts that this winding number is equal to the number of zeroes minus the number of poles.

There is one simple pole at $z = i$. Therefore

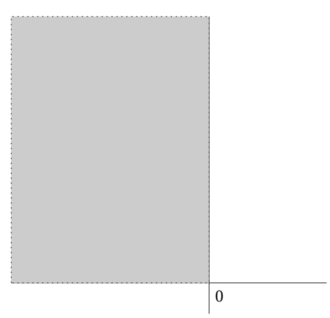
$$n \;=\; \text{number of zeros} \;-\; \text{number of poles} \;=\; \text{number of zeros} \;-\; 1\,.$$

There are $n+1$ zeros.

**Alternate.** $p(z) := (z-i)g(z) = (z-i)z^n + 1$ is a polynomial of degree $n+1$ so has exactly $n+1$ roots.

Since $p(i) = 1 \neq 0$ it follows that $p$ has exactly $n+1$ roots in $\mathbb{C} \setminus i$. Therefore $g$ also has $n+1$ roots in $\mathbb{C} \setminus i$. Since $i$ is not a root of $g$ this counts the roots of $g$. Note the care needed to treat the danger of division by zero when passing from $p$ to $g$.

**4.** (2+2 points). Define $\arg(z)$ in the slit plane $\mathbb{C}\backslash]-\infty, 0]$ by $-\pi \;<\; \arg(z) \;<\; \pi$. Define the analytic $\ln(z) = \ln|z| \;+\; i\arg(z)$ on the slit plane. Then $G(z) := \ln(z) - i\pi$ is analytic in the quadrant $\{z = x + iy \;:\; x < 0, \;\; 0 < y\}$. $G$ is continuous up to the boundary of the quadrant on the $x$-axis with real values there.



**i.** By what formula does $G(z)$ continue analytically to the left half plane $\{x < 0\}$?
**ii.** Compute the value of the continuation $G(-1-i)$.

**Solution. i.** In the left half plane the values of $G(z)$ in $\operatorname{Im} z < 0$ are computed from the values in $\operatorname{Im} z > 0$ from the formula

$$G(z) \;=\; \overline{G(\overline{z})}\,.$$

This identity then holds throughout the left half plane.
**ii.** Compute

$$\begin{aligned}\overline{G(-1-i)} \;&=\; G(\overline{-1+i}) \;=\; G(-1+i) \\ &=\; \ln(-1+i) - i\pi \;=\; \big((\ln\sqrt{2}) + i3\pi/4\big) - i\pi \\ &=\; \ln\sqrt{2} - i\pi/4..\end{aligned}$$

Therefore $G(-1-i) = \ln\sqrt{2} + i\pi/4$.

**5.** (6 points). Evaluate

$$\oint_{|z-3|=1} \frac{\cos z}{\sin z}\; dz\,.$$

**Solution.** The sin function in the denominator has simple zeros at the points $z = n\pi$ for integer $n$. The cos in the numerator does not vanish at these points so they are simple poles of $\cos z/\sin z$.

The contour encloses only the pole at $z = \pi$. The residue theorem implies that

$$\oint_{|z-3|=1} \frac{\cos z}{\sin z}\; dz \;=\; 2\pi i \operatorname{Res}\big(\cos z/\sin z\,,\; z = \pi\big)\,.$$

Since $\cos \neq 0$ at the simple poles, the residue is given by the value of $\cos(z)/\sin'(z)$ evaluated at $z = \pi$. Since $\sin' = \cos$, the residue is equal to 1.

**Ans.** $2\pi i$.

**Alternate.** The residue can be computed by expanding cos and sin in Taylor series about $z = \pi$. Those series are quickly computed using

$$\cos z = \cos((z-\pi)+\pi) = -\cos(z-\pi) = -\Big(1 - \frac{(z-\pi)^2}{2!} + \cdots\Big)\,,$$

$$\sin z = \sin((z-\pi)+\pi) = -\sin(z-\pi) \;= -\Big((z-\pi) - \frac{(z-\pi)^3}{3!} + \cdots\Big).$$

**6.** (8 points). Evaluate exactly

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\; dx\,.$$

*You may leave the answer as a complicated algebraic expression in complex numbers.*

**Solution.** The function

$$f(z) \;=\; \frac{z^2}{1+z^4}$$

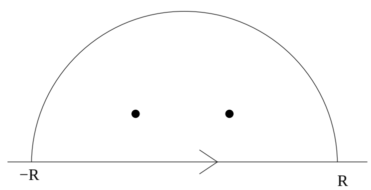
has poles at the roots of the denominator that are the fourth roots of -1, $(\pm 1 \pm i)/\sqrt{2}$. These poles are on the vertices of a square centered at the origin and sides parallel to the axes.

The integrand is infinitely differentiable on the $x$ axis and for $|z| > 1$,

$$|f(z)| \;\le\; \frac{|z|^2}{|z|^4 - 1}\,.$$

Thus the integral is absolutely convergent by comparison at $\pm\infty$ with the integral of $1/x^2$.

Apply the Residue Theorem to the open half disk $\mathcal{R} := \{|z| < R\,,\ \ \operatorname{Im} z > 0\}$ with $R > 1$.




The function $f$ is analytiic in $\mathcal{R}$ and on the boundary with the exception of isolated singularities at the poles $(\pm 1 + i)/\sqrt{2}$. The Residue Theorem implies that

$$\oint_{\partial\mathcal{R}} \frac{z^2}{1+z^4}\, dz \;=\; 2\pi i \Big(\operatorname{Res}(f\,,\,(1+i)/\sqrt{2}) \;+\; \operatorname{Res}(f\,,\,(-1+i)/\sqrt{2})\Big)\,.$$

Take the limit $R \to \infty$. On the circular boundary of length $\pi R$ one has $|f| \le R^2/(R^4-1)$ so the integral over that boundary tends to zero. The integral over the part of the boundary on the $x$-axis converges to the desired integral so

$$\text{Ans.} \;=\; 2\pi i \Big(\operatorname{Res}(f\,,\,(1+i)/\sqrt{2}) \;+\; \operatorname{Res}(f\,,\,(-1+i)/\sqrt{2})\Big)\,.$$

Let $g(z) = z^2$ and $h(z) = 1 + z^4$. Since $g$ is non zero and $h$ has a simple zero at each of the poles the residue at a root $\omega$ of $-1$ is given by

$$\frac{g(\omega)}{h'(\omega)} \;=\; \frac{\omega^2}{4\omega^3} \;=\; \frac{1}{4\omega} \;=\; \frac{\overline{\omega}}{4}\,,$$

since $|\omega| = 1$.

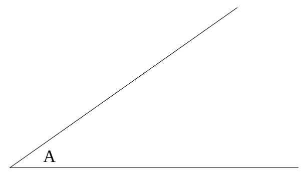
Therefore

$$\text{Ans.} \;=\; \frac{2\pi i}{4}\left(\frac{\overline{(1+i)}}{\sqrt{2}} + \frac{\overline{(-1+i)}}{\sqrt{2}}\right) \;=\; \frac{2\pi i}{4}\,\frac{(-2i)}{\sqrt{2}} \;=\; \frac{\pi}{\sqrt{2}}\,.$$

**7.** (5 points). Define the argument to have values in $]-\pi,\pi[$. The sector with angular opening $A$ is the set

$$\Big\{z\,:\,0<\arg z<A\Big\},\qquad 0<A<\frac{\pi}{2}\,.$$

A sketch follows.

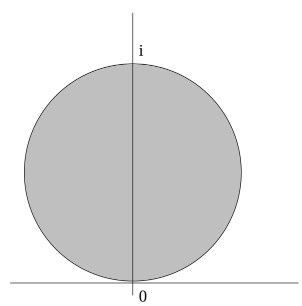



Find a one to one conformal mapping from the sector to an open disk.

**Solution.** The mapping $z\mapsto z^{\pi/A}$ maps the sector to the upper half plane.

The mapping $z\mapsto 1/(z-i):=k(z)$ maps the upper half plane to the inside of the circle that is the image of the real axis. That is a disk. That answers the question.

Continuing one could identify the circle by noting that it contains $0=k(\infty)$. And it contains the point $i=k(0)$. The mapping $k(z)$ preserves the imaginary axis. Since the imaginary axis is perpendicular to the $x$ axis their images are also perpendicular. Therefore the circle is perpedicular to the imaginary axis. The circle is $D:=\{|z-(i/2)|=1/2\}$ sketched below.



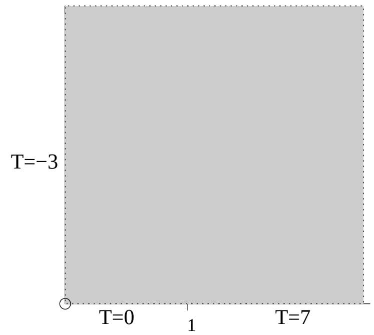

The mapping

$$F(z)\;=\;\frac{1}{z^{\pi/A}-i}$$

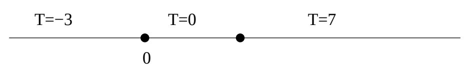
maps the sector conformally to the disk $D$.

**8.** (6 points). Find the unique bounded steady temperature distribution (a.k.a. harmonic function) in the postive quandrant $\{z=x+iy\,:\,x>0\;\;y>0\}$ with temperature equal to -3 on the strictly positive $y$-axis, and on the $x$-axis has teperatures equal to 0 on $]0,1[$

and equal to 7 on the segment $]1,\infty[$.




**Solution.** The map $z \mapsto z^2$ takes the quadrant problem to the correpsonding upper halfspace problem with boundary values sketched below.




Define the argument function to take values $0 \le \arg \le \pi$ in the upper half plane. Seek a solution of the form

$$A\arg(z-0)\ +\ B\arg(z-1)\ +C\,.$$

On the interval $]1,\infty[$ the two args vanish and one finds that

$$C = 7\,.$$

On the interval $]0,1[$ the first arg vanishes and one finds

$$B\pi + C = 0\,, \qquad B\ =\ \frac{-7}{\pi}\,.$$

On the interval $]-\infty,0[$ one finds

$$A\pi + B\pi + C = -3 \qquad A\ =\ \frac{-3}{\pi}\,.$$

The solution of the halfspace problem is

$$\frac{-3}{\pi}\arg(z-0)\ +\ \frac{-7}{\pi}\arg(z-1)\ +7\,.$$

A (and therefore the) solution of the quadrant problem is therefore

$$\textbf{Ans.}\ =\ \frac{-3}{\pi}\arg(z^2-0)\ +\ \frac{-7}{\pi}\arg(z^2-1)\ +7\,.$$

*For the problem 9 it may be useful to recall that* $(z+z^{-1})/2$ *is a conformal map from* $\{\mathrm{Im}\, z > 0, \quad |z| > 1\}$ *to the upper half space that leaves* $\pm 1$ *fixed and maps the intervals* $]-\infty, -1[$ *and* $[1, \infty[$ *to themselves.*

**9.** (5+1 points) **i.** Define the argument to have values in $]-\pi, \pi[$ and suppose that $0 < B < \pi$ Find a nonzero irrotational incompressible flow in the region

$$\Omega \;:=\; \Big\{ z \,:\, 0 < \arg z < B\,, \quad 1 < |z| < \infty \Big\}\,,$$

with flow parallel to the boundary at all boundary points.

**ii.** If $G(z) = \phi + i\psi$ is the complex potential, it is true that one of $\phi$, $\psi$ is constant on the boundary. Which one and why?

**Solution. i.** The map $z \mapsto z^{\pi/B}$ maps the region to the part of the upper halfspace exterior to the unit disk.

The map $z \mapsto (z+z^{-1})/2$ maps this region to the upper half space.

Therefore the map

$$H(z) \;:=\; \left( \frac{z^{\pi/B} + z^{-\pi/B}}{2} \right)$$

maps the sector to the upper half space.

The function $z$ is the complex potential of flow in the upper half space that it tangent to the boundaries. Therefore $H(z)$ is the complex potential of a flow in $\Omega$ that is tangent to the boundaries. If $H = \phi + i\psi$ the velocity field of the irrotational incompressible flow is

$$\nabla\big(\mathrm{Re}\, H\big) \;=\; \nabla \phi\,.$$

**ii.** The level sets of $\psi$ are streamlines. So the boundary is level set of $\psi$ when the boundary is connected.

**Discussion.** The velocity field constructed is **not** bounded. As in the Fluid Flow handout, one can show that there does not exist a bounded nonzero incompressible inrotational flow tangent to boundary.