Math 555. Fall 2013. Prof. J. Rauch. NAME ....

Midterm Exam October 17, 2013

Instructions. 1. Closed book. Two sides of a 3.5in. × 5in. sheet of notes from home.
2. No electronics, phones, cameras, . . . etc.
3. Show work and explain clearly.
4. There are 6 questions. They consist of 16 short subquestions each worth 5 points.
80 points total. You have about 5 minutes per short question. Be efficient.

1. (15 points). **i.** Find a polar form of the complex number $w = 1 + i$.

ii. If two students give correct answers $r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$, what can you say about the relation of the numbers r_1, r_2, θ_1 and θ_2?

iii. Find polar forms of $1/w$ and $\overline{w}$.

Solution. i. The modulus of w is equal to $\sqrt{2}$ and an argument is $\pi/4$ (45 degrees). A polar form is $w = \sqrt{2}e^{i\pi/4}$.

ii. Both the r_j must be equal to $\sqrt{2}$ and $\theta_1 - \theta_2$ must be an integer multiple of 2π.

iii. $1/w = (1/\sqrt{2})e^{-i\pi/4}$. $\overline{w} = \sqrt{2}e^{-i\pi/4}$.

2. (25 points). The function $f(z)$ is analytic on a disk $\mathbb{D}$ centered at the $z_0 = 2$, with Taylor series at the origin that begins with the three terms

$$f(z) \;=\; 1 + e\,(z-2) \;+\; \pi\,(z-2)^2 \;+\; \cdots\,.$$

i. Find the local expansion factor and rotation for the mapping $f(z)$ at $z_0 = 2$.

ii. Find the first three terms of the Taylor series of $f(z)^2$ centered at z_0.

iii. Find the first three terms of the Taylor series of $1/f(z)$ at $z_0 = 2$.

iv. Explain how you know that f is a one to one invertible map of a disk containing z_0 to an open set containing $w_0 = f(z_0)$.

v. Find the first three terms of the Taylor series at w_0 of the inverse function $f^{-1}(w)$.

i. From the expansion read off $f'(2) = e$. The local expansion factor is equal to $|f'(2)| = e$. The local rotation is by the argument of $f'(2)$, therefore zero.

ii.

$$f(z)\,f(z) \;=\; \Big(1 \;+\; e\,(z-2) \;+\; \pi\,(z-2)^2 \;+\; \cdots\Big)\Big(1 \;+\; e\,(z-2) \;+\; \pi\,(z-2)^2 \;+\; \cdots\Big)$$

Expand and collect terms of like powers yields for the first three terms

$$1 \;+\; 2e(z-2) \;+\; (e^2 \;+\; 2\pi)(z-2)^2 \;+\; \cdots\,.$$

Discussion. The first 25 points of this exam are VERY easy.

iii. Write

$$\frac{1}{f(z)} \;=\; \frac{1}{1 \;+\; e\,(z-2) \;+\; \pi\,(z-2)^2 \;+\; \cdots} \;=\; \frac{1}{1 + h(z)}\,,$$

with

$$h(z) \; := \; e\,(z-2) \; + \; \pi\,(z-2)^2 \; + \; \cdots .$$

Therefore

$$\frac{1}{f} \; = \; 1 - h + h^2 - h^3 + \cdots .$$

Since h^n has lowest power $(z-2)^n$ it suffices to consider up to h^2 to compute powers up to $(z-2)^2$. Compute

$$h^2 = \Big(e\,(z-2) \; + \; \pi\,(z-2)^2 \; + \; \cdots\Big)\Big(e\,(z-2) \; + \; \pi\,(z-2)^2 \; + \; \cdots\Big) \; = \; e^2(z-2)^2 + \cdots .$$

Therefore

$$\frac{1}{f} \; = \; 1 - e\,(z-2) - \pi\,(z-2)^2) + e^2(z-2)^2 + \cdots = 1 - e\,(z-2) + (-\pi + e^2)(z-2)^2 + \cdots .$$

An alternate, and shorter method is to posit

$$\frac{1}{f} \; = \; a_0 + a_1(z-2) + a_2(z-2)^2 + \cdots$$

and determine the coefficients by setting

$$1 + 0(z-2) + 0(z_2)^2 + \cdots \; = \; \Big(a_0 + a_1(z-2) + a_2(z-2)^2 + \cdots\Big)\Big(1 + 2e(z-2) + (e^2 + 2\pi)(z-2)^2 + \cdots\Big).$$

The terms of order 0, 1, 2 yield the trio of equations

$$a_0 = 1\,, \qquad a_0 e + a_1 = 0\,, \qquad a_0\pi + a_2 + a_1 e \; = \; 0\,,$$

that determine a_0, a_1, a_2.

iv. Since $f'(2) = e \neq 0$ the conclusion follows from the inverse function theorem.

v. Write $w = f(z)$, $z = f^{-1}(w)$. Denote $g := f^{-1}$, $w_0 = f(z_0) = 1$. Then

$$g(f(z)) = z$$

for z in a neighborhood of 2. Need $g(w_0), g'(w_0), g''(w_0)$ to get the first three terms of the Taylor series of g at $w_0 = 1$. Plug in $z = z_0$ to get

$$g(w_0) = z_0 = 2\,.$$

Differenitate with respect to z to find

$$g'(f(z))f'(z) \; = \; 1\,.$$

Plug in $z = z_0$ to find

$$g'(w_0))f'(z_0) = 1\,.$$

Therefore, $g'(w_0) = 1/e$. Differentiate a second time to find for all z near z_0,

$$g''(f(z))f'(z)f'(z) \; + \; g'(f(z))f''(z) \; = \; 0\,.$$

Use this identity at $z = z_0 = 2$. The value $f''(2) = \pi/2$ follows from the Taylor series. Together with previously determined quantities this yields

$$g''(w_0) \;=\; -2\pi/e^2\,, \qquad g(w) \;=\; 2 \;+\; (1/e)(w-1) \;-\; (\pi/e^2)(w-1)^2 \;+\; \cdots\,.$$

3. (10 points). **i.** Determine the order of the poles of

$$f(z) \;=\; \frac{z-1}{z^3-1}\,.$$

ii. Find the first two terms of the Laurent expansion of

$$g(z) = \frac{1}{z(z-1)}$$

valid in the annulus $\{z \,:\, |z| > 1\}$.

Solution. i. The poles can occur only at the roots of the denominator. The root $z = 1$ of the denominator is evident by inspection. Long division yields

$$z^3 - 1 \;=\; (z-1)(1+z+z^2)\,.$$

Thus

$$f(z) \;=\; \frac{1}{1+z+z^2}\,.$$

The denominator does not vanish at $z = 1$ so $z = 1$ *is not a pole.*

The quadratic formula shows that the denominator has two distinct simple roots

$$z_\pm \;=\; \frac{-1 \pm \sqrt{1-4}}{2}\,.$$

These are the cube roots of 1 other than 1. Since the roots are simple and the numerator is nowhere vanishing, the points $z_\pm$ are simple poles.

ii. Find the Laurent expansion (in powers of z^n) of $1/(z-1)$ valid in $|z| > 1$. Then divide by z. To find the Laurent expansion, factor z in the denominator to find

$$\frac{1}{z-1} = \frac{1}{z(1-(1/z))}\,.$$

In $|z| > 1$ one has $|1/z| < 1$ so the geometric series yields

$$\frac{1}{1-(1/z)} \;=\; 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\,.$$

The function f is $1/z^2$ times this so for $|z| > 1$,

$$f \;=\; \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\,.$$

4. (15 points). **i.** Evaluate

$$\oint_{|z-i|=10} \cos(z^2)\ (\sin z)^{1001}\ dz\,.$$

State clearly the theorem(s) you use.

ii. Evaluate

$$\oint_{|z-i|=10} \frac{\cos(z)}{(z-i)^{1001}}\ dz\,.$$

State clearly the theorem(s) you use.

iii. Evaluate

$$\oint_{|z|=10} \overline{z}\ dz\,,$$

where the circle is taken in the counterclockwise sense.

Solution. i. The integrand is analytic on $\mathbb{C}$. Cauchy's Theorem implies that for any domain R,

$$\oint_{\partial R} \cos(z^2)\ (\sin z)^{1001}\ dz \;=\; 0\,.$$

Applied to R equal to the disk $\{|z-i|<10\}$ yields the value 0 for the integral.

ii. If f is analytic in a region R as well as its boundary points then Cauchy's Integral Formula reads for all $z_0 \in R$,

$$f^n(z_0) \;=\; \frac{n!}{2\pi i}\oint_{\partial R} \frac{f(z)}{(z-z_0)^{n+1}}\ dz\,.$$

Apply with

$$f(z) = \cos(z)\,, \qquad n = 1000\,, \qquad R = \{|z|<10\}$$

to find that the integral is equal to

$$\frac{2\pi i}{1000!}\ \frac{d^{1000}\cos z}{dz^{1000}}\bigg|_{z=i}\,.$$

The 1000th derivative of cos is equal to cos because 1000 is a multiple of 4. So the integral is equal to

$$\frac{2\pi i}{1000!}\ \cos i\,.$$

Further simplification is possible.

iii. Parameterize $|z|=10$ by $\gamma(\theta) = 10\,e^{i\theta}$ with $0 \le \theta \le 2\pi$. By definition the integral is equal to

$$\int_0^{2\pi} \overline{\gamma(\theta)}\ \frac{d\gamma(\theta)}{d\theta}\ d\theta\,.$$

Since $\gamma' = 10ie^{i\theta}$ and $\overline{\gamma} = 10e^{-i\theta}$ this is equal to

$$\int_0^{2\pi} 10e^{-i\theta}\ 10ie^{i\theta}\ d\theta \;=\; 100i\int_0^{2\pi} d\theta \;=\; 200\pi i\,.$$

5. (10 points). **i.** Find the image of the rectangle

$$R \;:=\; \Big\{ 0 < \operatorname{Im} z < \pi\,, \quad 0 < \operatorname{Re} z < 1 \Big\}\,.$$

by the function e^z.

ii. Denote by $h(z)$ the branch of $z^{1/2}$ defined in the slit plane $\mathbb{C}\backslash] - \infty, 0]$ by $h(1) = 1$. Denote by arg the branch of the argument in the same set defined by $\arg(1) = 0$. Find the image by $h(z)$ of the sector

$$\Big\{ .1 < \arg(z) < .3\,, \quad 0 < |z| < 1 \Big\}\,.$$

Solution. i. For each $0 < x < 1$ consider the segment in R with real part equal to x. Its image is

$$e^{(x+iy)} \;=\; e^x e^{iy}\,, \qquad 0 < y < \pi\,.$$

The second factor traces out the upper half of the unit circle. The first factor is a real number ranging from 1 as x approaches 0 to e when x approaches 1.

As x varies these half circles sweep out the half annullus in the upper half plane between the circle of radius 1 and the circle of radius e.

ii. For each $.1 < \theta < .3$ the part of the ray with polar angle θ that lies in R is mapped to the ray with polar angle $\theta/2$ begining at the origin, the image of points near the origin, and ending with $r \to 1$ corresponding to the square roots of the points of modulus close to 1.

These images sweep out the sector

$$\Big\{ .1/2 < \arg(z) < .3/2\,, \quad 0 < |z| < 1 \Big\}\,.$$

6. (5 points). If v is a harmonic conjugate of u show that $-u$ is a harmonic conjugate of v.

Solution. Since v is a harmonic conjugate of u, the function $f := u + iv$ is analytic. Therefore $-if$ is analytic. Compute

$$-if = v - iu\,.$$

Therefore $-u$ is a harmonic conjugate of v.

Alternatively, one can verify the partial differential equations characterizing harmonic conjugates. Virtually all the exam papers took the alternate route.