Math 555, Fall 2013 Prof. J. Rauch

Homework 1. Due Thursday 12 September

There are three important pieces of background to buttress at the start of a complex variables course; ideas from the differential calculus of maps from the plane to itself, complex numbers, and concepts of convergence of sequences and series of functions.

The first problems of the first two assignments remind you how partial derivatives can be used to analyse the behavior of functions mapping the plane to itself. They have a computational and geometric side. The matrix of partial derivatives

$$J \;=\; \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

called the Jacobian matrix after Jacobi. For points x, y near $\underline{x}, \underline{y}$ one has

$$(\Delta u, \Delta v) \;\approx\; J(\underline{x}, \underline{y})\,(\Delta x, \Delta y)\,.$$

This fundamental relation is the two dimensional analogue of $\Delta y \approx f'(\underline{x})\Delta x$ that is the key relation in differential calculus of functions of one variable.

1. i. For inversion in the unit circle defined by

$$0 \;\neq\; (x,y) \quad\to\quad (u,v) \;:=\; \left(\frac{x}{x^2+y^2}\,,\,\frac{y}{x^2+y^2}\right)$$

compute the Jacobian matrix.

ii. For the mapping

$$(x,y) \;\mapsto\; (x-y\,,\,x^2+y^2),$$

compute the Jacobian matrix.

iii. Verify that for all (x,y) the Jacobian of the mapping in **i** is a matrix that maps circles to circles. Verify that for most (x,y) the Jacobian of the mapping in **ii** does not have this property.

Discussion. For **i** infinitesimal circles are sent to infinitesimal circles. That is a circle of radius $r << 1$ centered at a point $\underline{x}, \underline{y}$ is sent to a curve that is very close (error $\sim r^2$) to a circle centered at the image of $(\underline{x}, \underline{y})$. The images that might have been ellipses with any eccentricity, have eccentricity is equal to 1. For **ii**, the ellipses are nearly always noncircular.

2. 11/2a (This means problems 2a on page 11 of the text.)

3. 12/15b. **Hints.** Use polar form.

4. 12/20h.

5. 21/6. **Hint.** Use continuity of $0 \le x \mapsto \sqrt{x}$.

Continued on page 2.

6. 21/7. **Hints.** If D is a disk centered at α that does not reach the origin, find a single valued continuous function $\Theta(z) : D \to \mathbf{R}$ so that for each $z \in D$, the value $\Theta(z)$ is an argument of z. Use clear plain English to explain.

7. Carry out the following elegant proof that $e^{u+w} = e^u\, e^w$ for real u and w. The argument avoids manipulations of power series.

i. Show that $e^{-u}\, e^{u+w}$ is independent of u by computing its derivative with respect to u.

ii. Conclude that $e^{-u}\, e^{u+w} = e^w$.

iii. Use this for $w = 0$ and w to derive $e^{u+w} = e^u\, e^w$.

Discussion. 1. The proof extends immediately to the complex case once we define the complex derivative. **2.** This an example of a proof on a homework. It is just a series of computations. Nothing to provoke fear. **3.** This proof is very clever. It shows the beauty, elegance, and surprise that is possible in mathematical reasoning. **4.** The proof uses only the facts that the function $f(x) = e^x$ satisfies $f' = f$ and $f(0) = 1$.

The next exercise may be assigned in a future homework.

Exercise 1. There is an analogue of the preceding exercise for the logarithm. Show that if $g :]0, \infty[\to \mathbf{R}$ is a differentiable function satisfying $g'(x) = 1/x$ and $g(1) = 0$ then for all positive u and w one has $g(uw) = g(u) + g(w)$. **Hint.** Differentiate with respect to u.