Math 555, Fall 2013 **J. Rauch**
Homework 11 **Due Tuesday November 26**

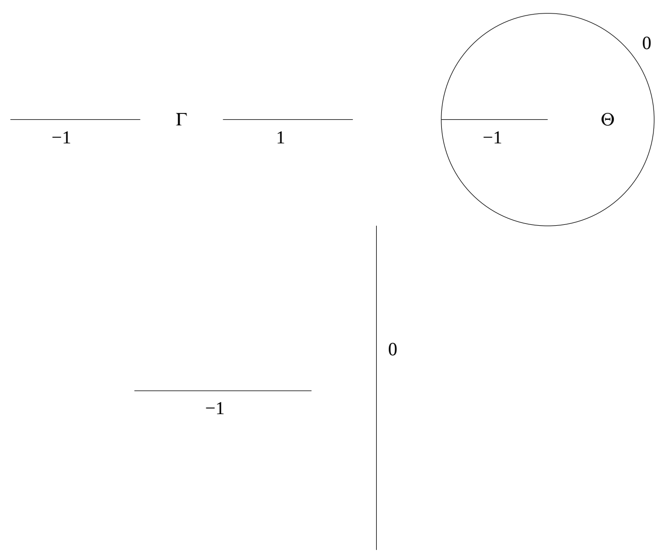
1. If F is a linear fractional transformation not equal to the identity show that there are at most two fixed points, that is values z so that $F(z) = z$.

2. Suppose that $-\infty < a < b < \infty$ and denote by $u(x, y)$ the unique bounded harmonic function in $y > 0$ that attains value 0 on $]-\infty, a[$, 1 on $]a, b[$, and, 0 on $]b, \infty[$. Find a harmonic conjugate of u in $y > 0$.

3. Find the unique steady state bounded temperature u in $y > 0$ that on the x-axis has value 0 on $]-\infty, -1[$, 1 on $]1, \infty[$, while the interval $]-1, 1[$ is insulated.

4. A steel plate is formed by cutting the unit disk into two pieces and discarding the lower piece. The cut is along the horizontal line through the point $z = e^{i\theta}$ with $0 < \theta < \pi$. The circular boundary of the resulting plate is kept at temperature $T = 1$. The horizontal boundary at $T = 0$. Find the unique bounded steady state temperature distribution.

5. 245/18 Variant. Denote by Γ the plane $\mathbb{R}^2$ with the segments $-\infty < x \leq -\sigma$ and $\sigma \leq x < \infty$ on the x-axis removed as in the figure. Find the unique temperature $u(x, y)$ harmonic and bounded on Γ with $u = -1$ on the left hand removed half line and $u = 1$ on the right hand removed half line. **Hint.** We know how to find a bounded harmonic function in the upper half disk that vanishes on the diameter and is equal to one on the top of the disk.



Consider the domain Θ equal to the unit disk with center at the origin with the interval $-1 < x \leq 0$ on the x-axis removed. The map $z^{1/2}$ suitably defined maps Θ to the right half disk. This allows one to solve the Dirichlet problem on the slit disk Θ with value -1 on both sides of the slit and value 0 on the circumference of Θ. Send the intersection of the slit with the circumference to infinity to solve a heat flow problem sketched in the third figure. Finish by using symmetry of the original problem with respect to the y-axis.

6. i. For $\varepsilon > 0$ denote by $P_{\pm} := \pm\varepsilon$. Find the unique bounded harmonic function defined in the upper half plane $y > 0$, continuous on $\{\mathrm{Im}\, z \geq 0\} \setminus \{P_{\pm}\}$ so that

$$\begin{aligned} &u(x,0) = 0 \quad \text{when} \quad -\infty < x < -\varepsilon, \\ &u(x,0) = 1 \quad \text{when} \quad \varepsilon < x < \infty, \\ &\frac{\partial u(x,0)}{\partial y} = 0 \quad \text{when} \quad -\varepsilon < x < \varepsilon\,. \end{aligned}$$

This is the steady state termperature distribution with semiiinfinite rays at termperatures 0 and 1 separated by an insulated boundary strip of length 2ε.

ii. Show that as $\varepsilon \to 0$ the solution tends to the corresponding problem without any insulating segment.

7. Find the unique bounded harmonic function in the upper half disk $\{|z| < 1\} \cap \{\mathrm{Im}\, z > 0\}$ that vanishes on the circular part of the boundary as well as on the boundary segment along the positive real axis and satisfies the Neumann boundary condition $\partial u/\partial y = 0$ on the boundary segment along the negative real axis.