Math 555, Fall 2013 **J. Rauch**
Homework 12 **Due December 3**

Because of the holiday weekend, this assignment will be accepted without penalty on Thursday Dec. 6. Assignment 13 is due on the normal date of Tuesday December 10.

1. 243/3,4 variant. **i.** Sketch the isotherms of the steady state heat flow described by the harmonic function $u = \operatorname{Re} f(z)$ where $f(z)$ is the function in 243/3.

ii. Sketch the isotherms of the steady state heat flow described by the harmonic function $u = \operatorname{Im} f(z)$ where $f(z)$ is the function in 243/3.

Discussion. When $f = u + iv$ is analytic then the level curves of u and the level curves of v are orthogonal as shown in a previous HW assignment. Thus the isotherms for each of these problems are the heat flow lives of the other.

2. 245/19, variant. The answer given is the unique bounded solution. There are many unbounded solutions. You are asked to use conformal mapping techniques to find the solution. Your answer need not look like the answer in the book.

3. Suppose that F is analytic in the wedge $0 < \arg z < A$ where $A < \pi$ and the branch of arg takes values in $]-\pi, \pi[$. Suppose in addition that for any $\sigma > 0$,

$$F(\sigma z) \;=\; c(\sigma)\, F\,, \tag{1}$$

for a suitable real $c(\sigma)$ depending on σ.

i. Show that if σ and τ are two positive constants then $c(\sigma\tau) = c(\sigma)c(\tau)$.

ii. Show that c is a differentiable function on $]0, \infty[$. **Hint.** Consider a single fixed z.

iii. Show that F is a homogeneous function of z. **Hint.** Use real logarithms to nearly determine $c(\sigma)$.

iv. Show an analytic F on the wedge that satisfies (1) if and only if it is of the form $b\, z^\alpha$ for some real α and complex b. **Hint.** Read the earlier homework problem identifying analytic functions homogeneous of degree n with n integer.

4.

5. 213/21. **Hints.** Use repeated reflection to extend the map to a map from the punctured plane $\mathbb{C} \setminus 0$ to itself. This takes an infinity of reflections. Then show that map satisfies $|F(z)| \le C\,|z|$. Conclude that the singularity at the origin is removable and then that $F(z) = az$ for some $a \in \mathbb{C}$. **Discussion.** This is a very important example. The Riemann Mapping Theorem in §13.34 shows that there is a conformal mapping from every simply connected domain strictly smaller than $\mathbb{C}$ to the unit disk. Thus any two such domains can be mapped one to the other. The problem shows that for non simply connected domains with the same number of holes, there is usually no such conformal mapping from one to the other.

6. Exercise 5.6 of the Neumann Problem handout.