Math 555 Fall 2013 Prof. J. Rauch
Homework 3 Due September 26

1. i. Suppose that $\Omega \subset \mathbb{C}$ is open, $f = u + iv : \Omega \to \mathbb{C}$ is analytic, $\underline{z} = \underline{x} + i\underline{y} \in \Omega$, and, $f'(\underline{z}) \neq 0$. Show that both $\text{grad}\, u(\underline{x}, \underline{y}) \neq 0$ and $\text{grad}\, v(\underline{x}, \underline{y}) \neq 0$.

The Implicit Function Theorem then implies that each of the level curves

$$\big\{(x,y)\ :\ u(x,y) = u(\underline{x},\underline{y})\big\} \qquad \text{and,} \qquad \big\{(x,y)\ :\ v(x,y) = v(\underline{x},\underline{y})\big\}$$

are smooth curves near $(\underline{x}, \underline{y})$. For example, if $u_y(\underline{x}, \underline{y}) \neq 0$ then the IFT implies that the level set of u is locally a smooth graph $y = h(x)$ with h infinitely differentiable. (Here we use the fact that u and v are infinitely differentiable.)

ii. Show that the level curves $u = u(\underline{x}, \underline{y})$ and $v = v(\underline{x}, \underline{y})$ are orthogonal at $\underline{z}$. **Hint.** Show that the normal vectors to the level curves are orthogonal. **Discussion.** If you graph the system of level curves of u and those of v you have curves that intersect at right angles. They define an orthogonal local coordinate system.

2. Show that $e^{\overline{z}}$ is not analytic. **Hint.** $e^x e^{-iy}$.

4. i. Show that if $m \neq n$ are integers then

$$\int_0^{2\pi} e^{im\theta}\ e^{-in\theta}\ d\theta \ =\ 0. \tag{1}$$

ii. With the same m, n evaluate

$$\int_C z^m\ \overline{z}^n\ dz$$

where C is the unit circle traversed in the positive sense.

Discussion. Identity (1) is important for Fourier series. It asserts that the functions $e^{im\theta}$ and $e^{in\theta}$ are orthogonal in the L^2 scalar product on 2π periodic functions. The scalar product is defined as

$$(g, h) \ =\ \int_0^{2\pi} g(\theta)\ \overline{h(\theta}\ d\theta\,.$$

5. 73/9.

6. 74/15.

7. 74/16. Add to parts **a,b,c** the next question. **d.** Use Cauchy's Theorem to explain why the sum of the first two must be equal to the third.