Math 555, Fall 2013 **J. Rauch**
Homework 8 **Due Tuesday November 5, 2013**

1. i. Show that the function $1/\sin z$ has simple poles at the points $n\pi$ with residue equal to $\cos n\pi = (-1)^n$.

ii. Show that $\cos \pi z/\sin \pi z$ has poles at the integers n with residues equal to $1/\pi$.

iii. Let C_N be the circle of radius $N + 1/2$ with center equal to the origin. Show that

$$\lim_{N\to\infty} \oint_{C_N} \frac{1}{z^2} \frac{\cos \pi z}{\sin \pi z}\, dz \;=\; 0.$$

2. iv. Compute

$$\text{Res}\left(\frac{1}{z^2}\frac{\cos \pi z}{\sin \pi z}\,,\, 0\right).$$

v. Apply the Residue Theorem to compute $\sum_{n=1}^{\infty} 1/n^2$.

Discussion. This is a broadly applicable method to sum infinite series using residues. It sums, for example, $\sum_{n=-\infty}^{\infty} P(n)/Q(n)$ for polynomials P, Q so that

- Q has no real roots,
- $\deg Q \geq \deg P + 2$, and,
- one knows exactly the roots of Q.

The problem is a case where Q does have a root on real axis.

3. Use the result of problem 6 of Homework 7 whose solution is posted on my office door, to compute using antiderivatives,

$$\int_0^{\infty} \frac{x}{1+x^3}\, dx.$$

Discussion. This is an example where the antiderivative works and the standard residue method does not. **Warning.** Be careful about branches of logarithms. You need them to be defined on the positive real axis.

4. Compute

$$\int_0^{\infty} \frac{x\ \sin 2x}{x^2+3}\, dx\,.$$

Ans. $(\pi/2)e^{-2\sqrt{3}}$.

Discussion. This integral is not absolutely convergent, but exists as an improper Riemannn integral.

5. Evaluate using residues

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x^3+1}{x^4+1}\, dx\,.$$

Hint. The example in class where the degree of P is one lower that the degree of Q is a model.

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7. Exercise on Homework 1.