Conformal Primer

Some students have difficulty getting the hang of the geometric side complex variables. This becomes fatal when one solves boundary value problems by conformal mappings. Here are a sequence of easy problems that introduce the basics. Once you get these you need to add the exponential function and its inverse to your arsenal. The linear fractional transformations are combinations of maps already on the list.

1. Define
$$A = \left\{ z \in \mathbb{C} \ : \ 0 < \arg z < a, \ \ |z| < 1 \right\}$$
where $0 < a < \pi/4$.

i. Sketch A.

Find and sketch the image of A under the following maps.

ii. $z \mapsto z + b$ with $b > 0$ real.

iii. $b < 0$, real.

iv. $b \in \mathbb{C} \setminus 0$.

v. $z \mapsto bz$ with $b > 0$ real.

vi. $b < 0$, real.

vii. $b \in \mathbb{C} \setminus 0$.

viii. $z \mapsto z^2$.

ix. $z \mapsto z^3$.

x. $z \mapsto z^4$.

2. Find and sketch the image under the map $z \mapsto 1/z$ of the following sets.

i. $\left\{ z \ : \ z/\underline{z} \in \mathbb{R}_+ \right\}$ with $\underline{z} \in \mathbb{C} \setminus 0$ fixed.

ii. $\left\{ z \ : \ z/\underline{z} \in]0,1[\right\}$ with $\underline{z} \in \mathbb{C} \setminus 0$ fixed.

iii. $\left\{ z \ : \ z/\underline{z} \in]1,\infty[\right\}$ with $\underline{z} \in \mathbb{C} \setminus 0$ fixed.

iv. The y-axis.

v. The x-axis.

vi. $\left\{ z \ : \ \operatorname{Re} z = 3 \right\}$.

vii. $\left\{ z \ : \ \operatorname{Im} z = 3 \right\}$.

viii. The circle $\left\{ z \ : \ |z| = 1/4 \right\}$.

ix. The circle $\left\{ z \ : \ |z| = 8 \right\}$.

x. The circle $\left\{ z \ : \ |z - 4| = 4 \right\}$.

xi. The circle $\left\{ z \ : \ |z - 4| = 1 \right\}$.