J. Rauch Math 555 Fall 2009

Conformal Matrices

Abstract We analyse the elliptical image of spheres by linear transformations. We characterize those transformations which preserve lengths (orthogonal matrices) and those that map spheres to spheres (conformal matrices). The Jacobian matrices of analytic functions are conformal and orientation preserving wherever they are invertible.

1 Transposes.

Denote the *standard scalar product* of vectors in $\mathbb{R}^n$ by

$$\langle x\,,\,y\rangle \;=\; \sum x_i\,y_i\,.$$

Suppose that A_{ij} is an $n \times n$ real matrix. The **transpose** A^t of A is defined by

$$(A^t)_{ij} \;:=\; A_{ji}\,.$$

The matrix of the transpose is the matrix of A flipped in the diagonal.

Example 1.1.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t \;=\; \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\,.$$

Proposition 1.2. *For all vectors x and y, $n \times n$ matrices A and B, and real numbers α,*

i. $(A + B)^t = A^t + B^t$,

ii. $(\alpha A)^t = \alpha A^t$,

iii. $(AB)^t = B^t A^t$,

iv. $\langle Ax\,,\,y\rangle \;=\; \langle x\,,\,A^t y\rangle$.

2 Length preserving linear transformations.

Theorem 2.1. *If A is a real $n \times n$ matrix then the following are equivalent.*

1. *For all x, $\|Ax\| = \|x\|$.*

2. *For all x, y, $\langle Ax\,,\,Ay\rangle = \langle x\,,\,y\rangle$.*

3. *A is invertible and $A^t = A^{-1}$.*

Definition 2.2. *The matrices satisfying these equivalent conditions are called* **orthogonal**.

Proof. It suffices to prove **1.** $\Rightarrow$ **2.** $\Rightarrow$ **3.** $\Rightarrow$ **1.**

(**1.** $\Rightarrow$ **2.**) Recall the algebraic identity for real numbers x and y,

$$xy \;=\; \big((x+y)^2 - (x-y)^2\big)/4\,.$$

Expanding as in elementary algebra shows that

$$\begin{aligned}\langle x+y\,,\,x+y\rangle \;&=\; \langle x, x+\rangle + \langle y, x+y\rangle \\ &=\; \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &=\; \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle\,,\end{aligned}$$

the last using the symmetry. Similarly

$$\langle x-y\,,\,x-y\rangle \;=\; \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \;=\; \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle\,,$$

Subtracting proves the *polarization identity*,

$$\langle x,y\rangle \;=\; \big(\langle x+y\,,\,x+y\rangle - \langle x-y\,,\,x-y\rangle\big)/4\,.$$

Property **i.** implies that the right hand side is equal to

$$\frac{\langle A(x+y)\,,\,A(x+y)\rangle \;-\; \langle A(x-y)\,,\,A(x-y)\rangle}{4}\,.$$

Simplifying then using the polarization identity again yields,

$$\frac{\langle Ax+Ay\,,\,Ax+Ay\rangle \;-\; \langle Ax-Ay\,,\,Ax-Ay\rangle}{4} \;=\; \langle Ax, Ay\rangle\,.$$

This completes the proof that $\langle x,y\rangle = \langle Ax, Ay\rangle$.

(**2** $\Rightarrow$ **3**). Using **2** and Propoition 1.2.iv. shows that

$$\langle x,y\rangle \;=\; \langle Ax, Ay\rangle \;=\; \langle A^tAx\,,\,y\rangle\,.$$

This shows that for all x, y,

$$\langle (A^tA - I)x\,,\,y\rangle \;=\; 0\,.$$

For each x this shows that $(A^tA - I)x$ is orthogonal to all y so must vanish. The identity $(A^tA - I)x = 0$ for all x is equivalent to $A^t = A^{-1}$.

(**3** $\Rightarrow$ **1**) Compute,

$$\langle Ax, Ax\rangle \;=\; \langle A^tAx\,,\,x\rangle \;=\; \langle x\,,\,x\rangle$$

where the last equality uses **3**. This proves **1.** $\square$

3 Positive symmetric matrices.

Definition 3.1. *A* **symmetric real matrix** *R is one for which $R_{ij} = R_{ji}$ for all i, j.*

It is a fundamental fact that for every such matrix there is a real orthogonal $\mathcal{O}$ so that $\mathcal{O}^{-1}R\mathcal{O}$ is a diagonal real matrix

$$\mathcal{O}^{-1}R\mathcal{O} \;=\; \mathrm{diag}\big\{\lambda_1, \dots, \lambda_n\big\}\,. \tag{3.1}$$

If $e_1, e_2, \dots, e_n$ is the standard orthonormal basis for $\mathbb{R}^n$ then $\mathcal{O}e_1, \mathcal{O}e_2, \dots, \mathcal{O}e_n$ is a new orthonormal basis consisting of eigenvectors of R with eigenvalues λ_j.

Definition 3.2. *A symmetric real R is* **positive** *when all the λ_j are strictly positive.*

The image by a positive symmetric R of the ball of radius 1 centered at the origin is an n dimensional ellipsoid with axes of length $2\lambda_j$ in the directions of the eigenvectors $\mathcal{O}e_j$.

Definition 3.3. *For a positive symmetric R as in* (3.1) *the square root $\sqrt{R}$ is defined as*

$$\sqrt{R} \;:=\; \mathcal{O}\,\mathrm{diag}\big\{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}\big\}\mathcal{O}^{-1}.$$

The square root is symmetric, positive and satisfies $(\sqrt{R})^2 = R$.

Proof. Let $D := \mathrm{diag}\big\{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}\big\}$. Compute

$$\sqrt{R}^{\,t} \;=\; \big(\mathcal{O}\,D\,\mathcal{O}^{-1}\big)^t \;=\; (\mathcal{O}^{-1})^t D^t \mathcal{O}^t \;=\; \mathcal{O}D\mathcal{O}^{-1} \;=\; \sqrt{R}\,,$$

the last using the fact that $\mathcal{O}^t = \mathcal{O}^{-1}$ and $D = D^t$. For the square compute

$$(\mathcal{O}\,D\,\mathcal{O}^{-1})^2 = \mathcal{O}\,D\,\mathcal{O}^{-1}\mathcal{O}\,D\,\mathcal{O}^{-1} = \mathcal{O}D^2\mathcal{O}^{-1}.$$

That this is equal to R follows from the definition of D and (3.1). $\square$

It is not hard to show that there is only one such positive square root so the definition is independent of the choice of $\mathcal{O}$. [1]

[1] If A is a matrix with $A^2 = R$, then $AR = RA = A^3$ so A commutes with R. If in addition, A is symmetric then there is a possibly different orthogonal $\mathcal{O}$ which *simultaneously* diagonalizes A *and* R. Therefore both A and R are of the form $\mathcal{O}(\text{diagonal})\mathcal{O}^{-1}$. To prove uniqueness it suffices to show that a positive diagonal matrix has a unique positive diagonal square root. That is easy.

4 Polar decomposition.

If M is invertible then $M\,M^t$ is symmetric since

$$(MM^t)^t \;=\; (M^t)^t M^t \;=\; MM^t.$$

For $x \neq 0$,

$$\langle M\,M^t x\,,\,x\rangle \;=\; \langle M^t x\,,\,M^t x\rangle \;=\; \|M^t x\|^2 \;>\; 0\,.$$

The fact that these expressions are all positive is equivalent to the positivity of the matrix $M\,M^t$.

Theorem 4.1. *If M is an invertible $n \times n$ matrix then there are uniquely determined positive symmetric R and orthogonal $\mathcal{O}$ so that $M = R\mathcal{O}$. One has*

$$R \;=\; \sqrt{MM^t}, \qquad \text{and} \qquad \mathcal{O} \;=\; (\sqrt{MM^t})^{-1} M\,. \tag{4.1}$$

This is called the *polar decomposition* of the matrix.

Proof. If $M = R\mathcal{O}$ with positive R and orthogonal $\mathcal{O}$, then

$$M\,M^t \;=\; R\mathcal{O}(R\mathcal{O})^t \;=\; R\,\mathcal{O}\,\mathcal{O}^t\,R^t \;=\; R^2\,,$$

the last step because $\mathcal{O}\mathcal{O}^t = I$ and $R = R^t$. Thus (4.1) is the only possible polar decomposition.

It remains to prove that this uniquely determined representation satisfies the conditions. It suffices to verify that the matrix $\mathcal{O} := \sqrt{M\,M^t}^{-1} M$ is orthogonal, that is $\mathcal{O}^t\mathcal{O} = I$. Compute,

$$\mathcal{O}^t\mathcal{O} \;=\; (\sqrt{M\,M^t}^{-1} M)^t\,R^{-1}\,M \;=\; M^t\,\sqrt{M\,M^t}^{-1}\,\sqrt{M\,M^t}^{-1}\,M\,.$$

Next use the easily proved fact that the inverse of the square root is the square root of the inverse, so

$$\sqrt{M\,M^t}^{-1}\,\sqrt{M\,M^t}^{-1} = \sqrt{(M\,M^t)^{-1}}\,\sqrt{(M\,M^t)^{-1}} = (M\,M^t)^{-1} = (M^t)^{-1}\,M^{-1}\,.$$

Inserting this in the preceding identity yields,

$$\mathcal{O}^t\mathcal{O} \;=\; M^t\,(M^t)^{-1}\,M^{-1}\,M \;=\; I\,,$$

completing the proof. $\square$

The polar representation allows one to describe precisely the images of spheres and balls by linear transformations.

Corollary 4.2. **i.** *If M is an invertible matrix then the image by M of the unit sphere is an ellipse whose principal axes have length $2\lambda_j$ where the λ_j are the eigenvalues of R in the polar decomposition $M = R\mathcal{O}$. The direction of the axes are the corresponding eigendirections.*

ii. *The image is a sphere if an only if $R = cI$ with $c > 0$ if and only if $M^t M = c^2 I$.*

Proof. **i.** Since $\mathcal{O}$ is orthogonal it maps the unit sphere to itself. Then R maps it to the ellipse with axes on eigendirections of R with length $2\lambda_j$.

ii. The image is a sphere if and only if the λ_j are all equal. Call the common value c. Then $R = cI$. Squaring this identity shows that it is equivalent to $M\,M^t = c^2 I$. □

Definition 4.3. *Matrices satisfying the equivalent conditions of* **ii.** *are called* **conformal**.

Problem. For the Jacobian computed in class

$$J = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix},$$

determine whether it maps circles to circles or to noncircular ellipses.

Solution. Compute

$$J\,J^t \;=\; \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \;=\; \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

Since this is not a multiple of the identity, J maps circles to noncircular ellipses. So the nonlinear map, maps small circles about $(1,1)$ to small noncircular ellipses. The eigendirections of $J\,J^t$ give the axes of the ellipses.

This example shows that using the results of this handout is very easy!

Problem. *Determine all conformal* 2×2 *matrices.*

Solution. Write the matrix as

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is conformal exactly when MM^t is a multiple of the identity. Compute

$$MM^t \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;=\; \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}.$$

The matrix is invertible and conformal if and only if

$$ac + bd = 0, \qquad a^2 + b^2 = c^2 + d^2.$$

The first condition asserts that $(c, d) \perp (a, b)$. Since (a, b) is nonzero by invertibility, this holds if and only if (c, d) is a multiple of $(-b, a)$. The second condition asserts that they have the same length. Therefore $(c, d) = \pm(-b, a)$ and the general solution is

$$\begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix}, \qquad a^2 + b^2 > 0\,. \tag{4.2}$$

Example 4.4. *If $b = 0$ and $a > 0$ the conformal matrices are*

$$\begin{pmatrix} a & 0 \\ 0 & \pm a \end{pmatrix}.$$

The first is a times the identity. The second a times reflection in the y-axis. They both map circles to circles. The second reverses orientation.

Theorem 4.5. *If M is linear and invertible from $\mathbb{R}^2$ to itself then the following are equivalent.*

1. *M is conformal and orientation preserving.*

2. *There is a nonzero complex number $\alpha + \beta i$ so that the the image by M of $x + iy$ is equal to $(\alpha + i\beta)(x + iy)$.*

The only linear conformal orientation preserving maps of $\mathbb{R}^2$ to itself are given by multiplication by complex numbers.

Proof. The determinant of the matrix (4.2) is equal to $\pm(a^2 + b^2)$. So the transformation is orientation preserving exactly when the determinant is positive which is the case $(c, d) = (-b, a)$. Thus, the most general orientation preserving invertible conformal transformation is

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad a^2 + b^2 > 0\,. \tag{4.3}$$

Expanding

$$(\alpha + i\beta)(x + iy) \;=\; (\alpha x - \beta y) + i(\beta x + \alpha y)$$

shows that the matrix of the linear tranfsormation $x + iy \mapsto (\alpha + i\beta)(x + iy)$ is equal to (4.3) when $\alpha = a$ and $\beta = -b$. This proves the result. $\square$