The Dirichlet Problem

Abstract. *We solve, by simple formula, the Dirichlet Problem in a half space with step function boundary data. Uniqueness is proved by complex variable methods. The formula for the Poisson kernel follows. The case of the disk follows by a fractional linear transformation.*

1. Introduction

For a nice bounded domain G and function $g : \partial G \to \mathbb{R}$ the **Dirichlet Problem** is to show that there is one and only one harmonic function $u : G \to \mathbb{R}$ so that the restriction of u to the boundary is equal to g. And to find as many interesting qualitative properties of the solution.

The classic formulation is to consider continuous functions g and seek solutions u that are continuous up to the boundary. We will be considering g that are continuous except for jump discontinuities in which case the continuity is only required up to closed subintervals disjoint from the discontinuity points of g.

That this problem has a unique solution is suggested by physical intuition as follows. Consider heat flow in G where the boundary is maintained at the time independent temperature g. It seems reasonable that the time independent boundary condition will drive the solution in the limit $t \to \infty$ to a time independent solution. That solution must then be a time independent solution of $u_t = \nu \Delta u$ that is equal to g on the boundary. Therefore a solution of the Dirichlet Problem. On physical grounds again, one expects that throughout G,

$$\min_{\partial G} g \;\le\; u \;\le\; \max_{\partial G} g\,. \tag{1.1}$$

For continuous solutions u on the closed domain this estimate was proved in class as a corollary to the mean value properties of analytic and harmonic functions in the plane. The fact that physical intuition leads to a provable qualitative property gives one confidence in the model and the intuition.

In addition, (1.1) implies the uniqueness of solutions u to the Dirichlet problem that are continuous up to the boundary. Indeed, if there were two such solution u_1 and u_2 then the difference $u = u_1 - u_2$ would be a harmonic function continuous up to the boundary and vanishing on the boundary. Then (1.1) implies that $u = 0$ proving uniqueness. This shows that no additional data need be prescribed in order to determine the steady state. Showing that arbitrary solutions tend to the steady state is properly the domain of a more detailed investigation of Partial Differental Equations.

The special case of the Dirichlet Problem in the disk can be attacked by introducing polar coordinates $u = u(r, \theta)$ and expanding the periodic function of θ in a Fourier series. This leads to an exact solution formula involving a Greens' function called the Poisson kernel. The key element is that the real and imaginary parts of z^n are harmonic showing that

$$r^n \, e^{\pm in\theta}, \qquad n = 0, 1, 2, \dots$$

are harmonic functions so by superposition

$$\sum_{-\infty}^{\infty} c_n \, r^{|n|} \, e^{in\theta}$$

is harmonic too. There is a separate handout discussing that approach from a complex variables point of view.

A similar strategy using Fourier Integrals works for G equal to the upper half space. In that case there is no uniqueness unless one prescribes the additional condition that g and the solution u are required to be bounded. In that case the estimate (1.1) involves sups and infs instead of maxs and mins.

This note attacks the half space Dirichlet Problem starting with a different set of exact solutions derived from $Arg(z)$. This naturally leads to solutions for data on the boundary consisting of functions that are piecewise constant. Pursing this line using purely complex methods leads to the Poisson kernel and also estimate (1.1).

2. Step data for the half plane and the disk

The Dirichlet problem in the half space $\{y > 0\}$ asks one to find a harmonic function u with $u|_{y=0} = g(x)$ with g a prescribed function.

Since the function y is harmonic with zero boundary value, given any solution the functions $u + Cy$ are also solutions. To guarantee uniqueness one requires that u is bounded in $\{y > 0\}$.

For any analytic $F = u + iv$, u and v are harmonic. For example the function $F(z) = \ln z$ with $-\pi < \arg < 3\pi/2$ yields the two harmonic functions $\ln r$ and $\arg(z)$ on the open half space $y > 0$. The function arg is bounded, that is

$$\sup_{y>0} |\arg z| < \infty \, .$$

The boundary value on the x-axis of $\arg z$ is the discontinuous function that is equal to 0 on $]0, \infty[$ and to π on $]-\infty, 0[$.

Exercise 2.1. *Show that the function* arg *is continuous, and in fact infinitely differentiable on* $\{y \geq 0\} \setminus 0$.

There is no value for the argument at the origin and it is best left undefined. The uniqueness part of the Theorem below shows that that is justifiable.

Definition 2.2. *A function* $g : \mathbb{R} \to \mathbb{R}$ *is called a* **step function** *when there is an* $n \geq 1$ *and subdivision of* $\mathbb{R}$ *into* $n+1$ *intervals* $-\infty < x_1 < x_2 < \cdots < x_n < \infty$ *so that the restriction of* g *to each open interval is constant.*

Example 2.3. *A constant function is a step function. It suffices to take* $n = 1$ *and place the division point* x_1 *arbitrarily.*

Theorem 2.4. *If* $g(x)$ *is a step function with discontinuities at* $\{x_j\}$*, then there is one and only one bounded harmonic function* u *in* $\{y > 0\}$ *that is continuous in* $\{y \geq 0\} \setminus \{x_j\}$ *and so that* $u(x,0) = g(x)$ *on* $\mathbb{R} \setminus \{x_j\}$*. In addition there are uniquely determined constants* a_j*,* $1 \leq j \leq n+1$*, so that*

$$u \;=\; a_1 \arg(z - x_1) \;+\; \cdots \;+\; a_n \arg(z - x_n) \;\;+\; a_{n+1}\,. \qquad (2.1)$$

Proof. **Uniqueness**. If u_1 and u_2 are solutions, denote by $u := u_1 - u_2$. Choose a harmonic conjugate v to u in the simply connected set $\{y > 0\}$ and define $F = u + iv$.

Exercise 2.5. *Show that* v *and therefore* F *is continuous on* $\{y \geq 0\} \setminus \{x_j\}$.

The Schwarz Reflection Theorem shows that defining $F(z) := F(z^*)^*$ when $\operatorname{Im} z \leq 0$ yields a bounded function that is analytic in $\mathbb{C} \setminus \{x_j\}$. Each of the excluded points is an isolated singularity. Abuse notation by denoting by F that analytic continuation.

Though the function u is bounded the harmonic conjugate will in general not be bounded. The function F has isolated singularitiwa at the points x_j. The real part of F is bounded. Both essential singularities and poles have real parts unbounded in any neighborhood of each singularity so the only possibility is a removable singularity.

Exercise 2.6. *Give details for the last sentence.*

Therefore defining F appropriately at each x_j yields a entire function with bounded real part. The function $e^{F(z)}$ is therefore bounded and entire. Liouville's Theorem implies that $e^{F(z)}$ is constant function. Therefore $F(z)$ is constant.

Exercise 2.7. *Give details for the last sentence.*

Therefore u is constant. Since $u = 0$ on $\mathbb{R} \setminus \{x_j\}$ it follows that $u = 0$ everywhere. This proves uniqueness.

Existence. We show that there is a choice of the constants a_j so that the formula (2.1) is a solution.

$\mathbb{R} \setminus \{x_j\}$ is the disjoint union of $n+1$ open intervals denoted I_j for $1 \leq j \leq n+1$ the $n+1$. Number so that I_{j+1} is to the right of I_j. Denote by g_j the values of g on the corresponding interval. The key observation is that

$$\arg(z - x_j) \;=\; 0, \qquad \text{on} \qquad I_k \quad \text{for} \quad k \geq j\,.$$

Thus (2.1) holds if and only if

$$\pi(a_j + a_{j+1} + \cdots + a_n) \;+\; a_{n+1} \;=\; g_j, \qquad 1 \leq j \leq n+1\,.$$

The case $j = n+1$ holds if and only if

$$a_{n+1} = g_{n+1}.$$

Then the case n holds if an only if

$$\pi\, a_n + a_{n+1} \;=\; g_n$$

detemining a_n. Continuing, the values of the a_j are uniquely determined so that the function (2.1) satisfies the boundary value problem. □

Example 2.8. *If $g(x)$ is the characteristic function of $]0, \infty[$ (a.k.a. the heaviside function), the solution is*

$$u = 1 - (\arg z)/\pi\,. \tag{2.2}$$

Exercise 2.9. *Explain why the unique determination of the a_j does NOT prove uniqueness of the solution.*

Exercise 2.10. *Find the solution so that for $x \in]j-1, j[$ with $1 \leq j \leq N$, $u(x, 0) = j$, and, $u(x, 0) = 0$ otherwise. Sketch the boundary values.*

Exercise 2.11. *Denote by τ_h the operator that translates a function h units in the x direction*

$$(\tau_h u)(x, y) \;:=\; u(x - h, y)\,.$$

For $a \in \mathbb{R}$ and $h > 0$, show that the unique solution with g equal to the characteristic function of the interval $]a, a+h[$ is equal to

$$-\;\frac{\tau_{-h}(\arg(z-a)) \;-\; \arg(z-a)}{\pi}\,.$$

Conclude that the solution is strictly positive in the open upper half plane.

Exercise 2.12. *If g is a nonnegative step function not identically equal to zero, show that the solution is strictly positive in the open upper half plane.* **Hint.** Use the preceding exercise. **Discussion.** This recovers Estimate 1.1. The present proof does not depend on the maximum and minimum theorems for harmonic functions that depended on continuity up to the boundary. The present solutions are not continuous up to the boundary.

Exercise 2.13. *Show that the solution constructed in the Theorem takes values for $y > 0$ strictly between the minimum and maximum values of $g(x)$.* **Hint.** Use the preceding exercise.

Suppose now that $g(x)$ is continuous on $\mathbb{R}$ and that the two limits $\lim_{x\to\pm\infty} g(x)$ exist. Using the solvability of the Dirichlet problem for step function data, and approximating g uniformly by such data, it is not hard to prove that the Dirichlet problem with boundary value g has a unique bounded solution. A alternative strategy is given in the next section.

Corollary 2.14. *Suppose that $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ and $P_j := e^{i\theta_j}$ the finite set of corresponding points on the unit circle. Suppose that $g : S^1 \setminus \{P_j\}$ is constant on each of the intervals in $S^1 \setminus \{P_j\}$. Then there is one and only one harmonic function $u : \{|z| < 1\}$ that is uniformly bounded and continuous on $\{|z| \leq 1\} \setminus \{P_j\}$ and equal to g on $S^1 \setminus \{P_j\}$.*

Proof. Choose a fractional linear transformation $L(\zeta)$ conformally mapping the upper half plane to the unit disk. Then u solves the problem of the Corollary if and only if $w(\zeta) := u(L(\zeta)$ solves a Dirichlet problem of the same type for the upper half plane. Existence and uniqueness follow from Theorem 2.4. □

3. The Green's function for the upper half space

3.1. **Elementary version.** Suppose that $h > 0$ and $g = \mathbf{1}_{]\mathbf{a},\mathbf{a}+\mathbf{h}[}$ is the characteristic function of an interval. Exercise 2.11 shows that the solution of the Dirichlet Problem is

$$u(x, y) \;=\; \frac{\theta(x+h, y) - \theta(x, y)}{\pi}\,.$$

The Fundament Theorem of Calculus implies that with

$$G(x,y) \;:=\; -\,\frac{1}{\pi}\,\frac{\partial\theta}{\partial x}\,,$$

$$u(x,y) \;=\; \int_a^{a+h} G(x,y)\;dy \;=\; \int_{-\infty}^{\infty} G(x-s,y)\,g(s)\;ds\,.$$

To compute a formula for G, differentiate using $\theta = \text{Im}\,\ln z$. Therefore

$$\frac{\partial\theta}{\partial x} \;=\; \text{Im}\,\frac{\partial \ln z}{\partial x} \;=\; \text{Im}\,\frac{\partial \ln z}{\partial z} \;=\; \text{Im}\,\frac{1}{z} \;=\; \text{Im}\,\frac{\overline{z}}{|z|^2} \;=\; \frac{-y}{x^2+y^2}\,.$$

Therefore

$$G(x,y) \;=\; \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

The resulting formula

$$u(x,y) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-s)^2+y^2}\;g(s)\;ds$$

for the bounded solution of the Dirichlet Problem in the half space is called **Poisson's Integral Formula**. It works for for example when $g(x)$ is continuous tends to zero as $|x| \to \infty$.

3.2. **Version using Dirac's delta function.** There is an alternate derivation of the formula for $G(x,y)$ namely to solve the Dirichlet problem with initial temperature equal to 0 for $x<0$ and $x>\varepsilon$ and temperature $1/\varepsilon$ on $]0,\varepsilon[$. This initial temperature converges to $\delta(x)$ as $\varepsilon \to 0$. Call the solution $u_\varepsilon(x,y)$. Then

$$G(x,y) \;=\; \lim_{\varepsilon\to 0}\; u_\varepsilon(x,y)\,.$$

To see that the formulas are the same, define θ to be the standard argument with values in $]-\pi,\pi[$. Then Exercise 2.11 shows that

$$u_\varepsilon(x,y) \;=\; \frac{\theta(x-\varepsilon,y)-\theta(x,y)}{\varepsilon\,\pi}\,.$$

Therefore

$$\lim_{\varepsilon\to 0}\; u_\varepsilon(x,y) \;=\; -\,\frac{1}{\pi}\;\frac{\partial\theta(x,y)}{\partial x}\,.$$

This agrees with the formula from the preceding subsection.

4. Thermal equilibrium and electrostatics

When $F(z) = u+iv$ is analytic and u represents temperature at thermal equilibrium, the level curves of u are *isotherms* and $-\nabla u$ is orthogonal to those curves and is the direction of heat flow. The level curves of v, that are orthogonal to those of u, therefore represent *heat flow lines.*

If the level curves of v are sketched with equal increments in the constant values, then the density of level lines crossing a surface measures the flux of heat per unit time.

If u is the electrostatic potential then ∇u is the electric field so the level curves of u are orthogonal to the field. Thus the level curves of v are parallel to the field. They are the *field lines.* If the level curves of v are sketched with constant increments, then the density of lines represents the electric field strength.

The energy density for the heat flow problems is $u\, dxdy$ while in electrostatics it is $|E|^2\, dxdy = |\nabla u|^2\, dxdy$. For heat flow the harmonic function $\arg z$ has finite energy in each bounded set. *For electrostatics the energy in any bounded set containing* 0 *is infinite.* If one tries to arrange a laboratory realization of an electrostatic experiment for the solutions of section 1, there will be spectacular arcing from the adjacent intervals at different potentials. These boundary values are unattainable.

On the other hand, intervals of different electrostatic potential separated by an insulator is easily realized and the corresponding boundary value problems solved using the conformal mapping $\sin z$. In contrast to the results of §1 these interesting solutions do not generalize to simple formulas for more than two intervals of insulator.

5. Related examples

Uniqueness for some other boundary value problems can be proved using reflection. The next result is applied to the difference of two solutions.

Theorem 5.1. *If u is a uniformly bounded and continuously differentialbe function on $0 \le y \le L$ which is harmonic in $0 < y < L$ satifies $u(x,0) = 0$ and either*

$$\text{for all } x, \quad u(x,L) = 0 \qquad \text{or,} \qquad \text{for all } x, \quad \frac{\partial u(x,L)}{\partial y} = 0\,,$$

then $u = 0$.

Proof. First treat $u(x, L) = 0$. Choose a harmonic conjugate v in $0 \leq y \leq L$ and let $F(z) = u + iv$. Shwartz reflection across $y = L$ yields an analytic function in $0 < y < 2L$ with $u(x, 2L) = 0$ and the same uniform bound on u as in the initial strip. Repeating extends to $0 < y < \infty$ with the same uniform bound on u. A final relfection in $y = 0$ yields an entire F with uniformly bounded u. Therefore F is constant so u is constant. Since $u = 0$ on the x-axis it follows that $u = 0$.

For the case $u_y(x, L) = 0$ the Cauchy-Riemann equations imply that $v_x(x, L) = 0$ so v is constant on $y = L$ Adding a constant to v we may assume that $v(x, L) = 0$. A Shwartz reflection across $y = L$ then yields an analytic continuation to $0 < y < 2L$ with the same bound on u and $u(x, 2L) = 0$. The first result then applies to prove that $u = 0$. □

This result treats the case of the electric field between two infinite capacitor plates. One of the important examples not covered is the field describing edge effects in a semi-infinite or finite capacitor. These may be included in a future version.