The Dirichlet Problem in the Disk

Summary. *The Dirichlet Problem in the disk is solved two ways. The first uses the real and imaginary parts of z^n together with Fourier series. The second constructs the Poisson kernel by exactly solving for step data approaching Dirac's delta. Conformal map to the upper half plane to solve the step data problem.*

1. Solution by Fourier series.

The Dirichlet problem in the disk asks to find a bounded harmonic function u in the disk $\mathbb{D} := \{|z| < 1\}$ that assumes prescribed values $u(e^{i\theta}) = f(\theta)$ on the boundary. The periodic function f is given.

The maximum principal for harmonic functions implies that there is at most one such solution.

Complex function theory aids by providing many harmonic functions. Since z^n for $n \in \mathbb{N}$ is analytic its real and imaginary parts, $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic. Therefore by reflection in the x-axis, $r^n \cos(-n\theta)$ and $r^n \sin(-n\theta)$, are also harmonic. Taking linear combinations implies that the complex valued functions $r^n e^{in\theta}$ and $r^n\, e^{-in\theta}$ are harmonic for $n \in \mathbb{N}$.

Therefore for complex c_n that decay fast enough to guarantee convergence,

$$u \;=\; \sum_{n=-\infty}^{\infty} c_n\, r^{|n|}\, e^{in\theta} \tag{1.1}$$

is harmonic. The boundary values at $r = 1$ are given by the Fourier series $\sum c_n\, e^{in\theta}$.

Theorem 1.1. *If $f \in C^\infty_{periodic}$ then the unique solution of the Dirichlet problem is given by (1.1) with c_n the Fourier coefficients of f,*

$$c_n \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\; e^{-in\theta}\; d\theta\,. \tag{1.2}$$

Exercise 1.2. *Consider the solution of the Dirichlet problem in $\mathbb{D}$ with boundary value $1/(1 + \varepsilon \sin\theta)$ with $|\varepsilon| < 1$. Compute the first three terms in the perturbation series*

$$u \;\approx\; u_0(x,y) \;+\; \varepsilon u_1(x,y) \;+\; \varepsilon^2 u_2(x,y) \;+\; \cdots\;.$$

2. The Poisson kernel.

The formulas (1.1) (1.2) yield

$$u \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_n r^{|n|}\; e^{in\theta}\; f(\phi)\; e^{-in\phi}\; d\phi\,.$$

Define the **Poisson kernel**

$$P(r,\psi) \;:=\; \frac{1}{2\pi}\;\sum_n r^{|n|}\; e^{in\psi}\,.$$

Then,

$$u(r,\theta) \;=\; \int_{-\pi}^{\pi} P(r,\theta-\phi)\; f(\phi)\; d\phi\,. \tag{2.1}$$

Compute P in two pieces, each a geometric series. With $\beta := r\,e^{i\psi}$,

$$\sum_0^\infty r^{|n|}\; e^{in\psi} \;=\; \sum_0^\infty\; \beta^n \;=\; \frac{1}{1-\beta}\,,$$

$$\sum_{-\infty}^{-1} r^{|n|}\; e^{in\psi} \;=\; \sum_{-\infty}^{-1}\; \beta^n \;=\; \sum_1^\infty\; \overline{\beta}^n \;=\; \frac{\overline{\beta}}{1-\overline{\beta}}\,.$$

Adding yields

$$2\pi P = \frac{1}{1-\beta} + \frac{\overline{\beta}}{1-\overline{\beta}} = \frac{(1-\beta)+\overline{\beta}(1-\beta)}{(1-\beta)(1-\overline{\beta})} = \frac{1-r^2}{|1-\beta|^2} = \frac{1-r^2}{|1-r\,e^{i\psi}|^2}\,. \tag{2.2}$$

3. The Poisson kernel from conformal mappling.

Compute $P(r,\psi)$ by solving exactly the Dirichlet problem with boundary data f_ε a sequence of step functions converging to the Dirac delta. This follows the strategy used to compute the Poisson kernel for a half space in the handout on the Dirichlet problem.

Take f_ε equal to $1/2\varepsilon$ on an interval of length 2ε centered at the point 1 on $\partial\mathbb{D}$, and, $f_\varepsilon = 0$ outside this interval. Then (2.1) implies that if u_ε is the solution one has $P(r,\psi) = \lim_{\varepsilon\to 0} u_\varepsilon(r,\psi)$.

To find u_ε map the Dirichlet problem in the disk to one in the upper half space. Since the boundary datum is a step function the latter problem is exactly solvable.

To map to the half space one can proceed in small steps and then combine or can reason as follows.

Exercise 3.1. *Show that the fractional linear transformation F that satisfies*

$$F(1) \;=\; 0, \qquad F(-1) \;=\; \infty, \qquad F'(1) \;=\; -i$$

is a one to one conformal map of $\mathbb{D}$ onto the upper half space. **Hint.** First find the image of $\partial\mathbb{D}$.

To find F start with the observation that the fractional linear transformations satisfying $F(1) = 0$ and $F(-1) = \infty$ are of the form $F = C(z-1)/(z+1)$. The product rule for derivatives yields

$$F' \;=\; C\Big[(z-1)\frac{d}{dz}\frac{1}{z+1} \;+\; \frac{1}{z+1}\Big].$$

Setting $z = 1$ yields $F'(1) = C/2$. Thus $F'(1) = -i$ if and only if $C = -2i$ so

$$F(z) \;=\; -2\,i\,\frac{z-1}{z+1}\,. \tag{3.1}$$

The image by F of the arc connecting $e^{-i\varepsilon}$ to $e^{i\varepsilon}$ on $\partial\mathbb{D}$ is the interval $[-w_\varepsilon, w_\varepsilon]$ on the x-axis where $w_\varepsilon := F(e^{i\varepsilon})$. Taylor expansion yields

$$\begin{aligned} w_\varepsilon \;&=\; F\big(1 + i\varepsilon + O(\varepsilon^2)\big) \;=\; F(1) + F'(1)\,i\varepsilon + O(\varepsilon^2) \\ &=\; 0 \;+\; (-i)(i\varepsilon) \;+\; O(\varepsilon^2) \;=\; \varepsilon \;+\; O(\varepsilon^2)\,. \end{aligned} \tag{3.2}$$

Define g_ε to be the unique bounded harmonic function in the upper half plane with boundary value equal to $(2\varepsilon)^{-1}\chi_{[-w_\varepsilon, w_\varepsilon]}(x)$.[1] Then the exact solution u_ε is given by

$$u_\varepsilon(z) \;=\; g_\varepsilon(F(z))\,. \tag{3.3}$$

From the first handout on the Dirichlet problem, the solution g^ε is given by

$$g_\varepsilon(w) \;=\; \frac{\arg(w - w_\varepsilon) \;-\; \arg(w - (-w_\varepsilon))}{2\,\pi\,\varepsilon}\,. \tag{3.4}$$

To compute the limit of u_ε we compute the limit of g_ε. Thanks to (3.2)

$$\begin{aligned} \lim_{\varepsilon\to 0}\; g_\varepsilon(w) &= \frac{1}{-\pi}\,\frac{\partial}{\partial x}\,\arg w = \frac{1}{-\pi}\,\frac{\partial}{\partial x}\,\mathrm{Im}\,\ln w \\ &= \frac{1}{-\pi}\,\mathrm{Im}\,\frac{\partial}{\partial x}\,\ln w = \frac{1}{-\pi}\,\mathrm{Im}\,\frac{\partial}{\partial w}\,\ln w = \frac{1}{-\pi}\,\mathrm{Im}\,\frac{1}{w}\,. \end{aligned} \tag{3.5}$$

[1] Since g_ε is continuous in $\mathrm{Im}\, z > 0$ and the boundary values are discontinuous, the $\lim_{y\to 0} g_\varepsilon(x,y)$ is not uniform. It is uniform on closed bounded subsets of $\mathbb{R}_x \setminus \{\pm w_\varepsilon\}$. Together with boundedness, that guarantees uniqueness. Similarly $\lim_{r\to 1} u^\varepsilon(r,\theta)$ is uniform on closed bounded sets of θ in the complement of $\{\pm\varepsilon\}$. With boundedness this implies uniqueness (see exercise 3.3).

Therefore

$$\lim_{\varepsilon\to 0}\ g_\varepsilon(F(z)) \;=\; \frac{1}{-\pi}\ \mathrm{Im}\,\frac{1}{F(z)} \;=\; \frac{1}{-\pi}\ \mathrm{Im}\Big(\frac{1}{-2i}\frac{z+1}{z-1}\Big)$$
$$= \frac{1}{2\pi}\ \mathrm{Im}\Big(\frac{1}{i}\,\frac{z+1}{z-1}\Big) \;=\; \frac{-1}{2\pi}\ \mathrm{Re}\Big(\frac{z+1}{z-1}\Big)\,. \tag{3.6}$$

Therefore

$$P(r,\psi) \;=\; \frac{-1}{2\pi}\ \mathrm{Re}\Big(\frac{z+1}{z-1}\Big)\,, \qquad z = r\,e^{i\psi}\,. \tag{3.7}$$

Exercise 3.2. *Verify that* (3.7) *agrees with* (2.2)*.*

Exercise 3.3. *Prove the following uniqueness theorem.*

Theorem 3.4. *If u is a uniformly bounded harmonic function infinitely differentiable on the open disk $\mathbb{D}$ so that for almost all θ in the sense of Lebesgue measure*[2]

$$\lim_{r\to 1}\ u(r\,e^{i\theta}) \;=\; 0\,.$$

Then u is identically equal to zero on $\mathbb{D}$.

Hints. For $0<\rho<1$ define a smooth harmonic function on the closed disk by

$$v_\rho(x,y) \;=\; u(\rho\,x,\rho\,y)\,, \qquad x+iy \;=\; r\,e^{i\phi}\,.$$

Apply (2.2) to find

$$v_\rho(r\,e^{i\theta}) \;=\; \int_{-\pi}^{\pi} P(r,\theta-\phi)\ u(\rho\,e^{i\phi})\ d\phi\,.$$

For $x^2+y^2<1$ justify passage to the limit $\rho\to 1$ using the dominated convergence theorem.

[2]Without using measure theory one has same conclusion under the stronger hypothesis that one has uniform convergence on closed subsets of the complement of a finite set $\{\theta_1,\theta_2\cdots,\theta_N\}$. This hypothesis suffices to prove uniqueness of the solutions u_ε.