Fluid Flows and Complex Analysis

Summary. *Some classic and not so classic examples of incompressible, irrotational, planar flows are presented. Uniqueness and boundedness questions related to the examples are discussed using beautiful complex variable methods. The latter are absent in traditional presentations.*

1. Simplest flows.

For an incompressible irrotational velocity field $(u(x,y), v(x,y))$ the function

$$f \ := \ u - iv$$

is analytic. On a simply connected fluid domain there is an analytic antiderivative $F = \phi + i\psi$ with

$$F' \ = \ f\,.$$

F is called a **complex potential**. The Cauchy-Riemann equations show that ϕ is a **velocity potential** in the sense that

$$\nabla\phi \ = \ (u,v)\,.$$

Since F is analytic, the level curves of ψ are orthogonal to the level curves of ϕ. Since ϕ is a velocity potential the level curves of ϕ are orthogonal to (u,v). Therefore the level curves of ψ are parallel to (u,v) so are integral curves of the vector field (u,v). They are called **particle paths** and also **streamlines**.

Example 1.1. *The simplest flow has $F(z) = constant$ and zero velocity.*

Example 1.2. *The next simplest flow has complex potential $F(z) = z$. Then $F' = 1 + 0i$ so the velocity is $(1,0)$. The flow is parallel to the x-axis at constant speed. The streamlines are level sets of $\operatorname{Im} F = y$.*

Example 1.3. *The next simplest flow has complex potential $F(z) = z^2$. The streamlines are the level sets of $\operatorname{Im} F = 2xy$. They are hyperbolas with asymptotes the x and y axes. The flow is parallel to both the x and the y-axes.*

Example 1.4. *The example $F(z) = \log z = \log|z| + i \arg z$ yields interesting flows. F is not defined at the origin and is only defined up to additive constants of $2\pi i n$. But the velocity*

$$\overline{F'} \ = \ \frac{1}{\bar z} \ = \ \frac{z}{|z|^2} \ = \ \frac{x+iy}{x^2+y^2}$$

is independent of the additive constant. The stream lines are the level sets of $\operatorname{Im} F = \arg z$ *so are rays through the origin. The flow is outward from the origin. The speed is* $|F'| = 1/r$. *The outward flux through every circle centered at the origin is* 2π. *The flow is a source of strength* 2π *at the origin. The divergence of the flow is* 2π *times the Dirac delta at the origin. The fluid equations are not satisfied at that point.*

Example 1.5. *The potential* $F = i \log z$ *The streamlines are the level sets of* $\operatorname{Im} F = \log |z|$. *The streamlines are circles through the origin. This is a swirling flow. The speed is* $|F'| = 1/r$. *The circulation around any circle through the origin is equal to* 2π. *The curl of the flow is equal to* 2π *times the Dirac delta at the origin. The fluid equations are not satisfied at that point.*

2. Flows past the boundary of a half space.

Seek flow in the upper half space $\{y \geq 0\}$ that is tangent to the boundary. It represents flow past the x-axis that bounds the flow domain.

The second third and fourth flows of the last section are such flows. Linear combinations yield nontrivial examples that are all distinct. *There are even more.*

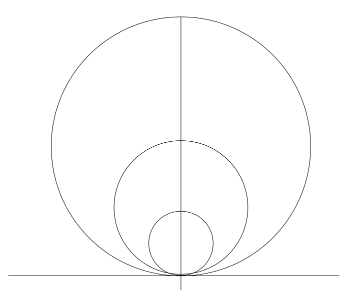
Example 2.1. *Consider the flow with potential*

$$F(z) \;=\; \frac{1}{z} \;=\; \frac{x - iy}{x^2 + y^2}$$

The x *axis is a level set* $\operatorname{Im} F = 0$ *so is a stream line.*

The streamlines are the curves $\operatorname{Im}(1/z) = \text{constant}$. Since $1/z$ preserves lines and circles, the streamlines are lines or circles.

Exercise 2.2. *Show that the streamlines are circles whose diameters lie on the* y*-axis. And the circles are tangent to the* x*-axis at the origin.*



This speed $|F'| = 1/|z|^2$ diverges as one approaches the origin. The velocity on the x axis is equal to $-1/x^2 < 0$. The fluid between two circles swirls counterclockwise and squeezes through at the origin while accelerating to infinitely large velocity. A finite volume squeezes through an infinitesimal space in finite time by going infintly fast. The next result shows that the flows with constant velocity parallel to the x-axis are the only ones with bounded velocity.

Theorem 2.3. *Suppose that $F(z) = \phi + i\psi$ is analytic and in $\{y > 0\}$, continuously differentiable in $\{y \geq 0\}$, and, with $\overline{F'}(z)$ parallel to the x-axis when $y = 0$. If F' is uniformly bounded in $\{y > 0\}$, then $F = az+b$ for some $a \in \mathbb{R}$, $b \in \mathbb{C}$.*

Proof. The x-axis is a particle path hence a level curve of ψ. Denote by c the value of ψ on this axis.

Then, $F - ic$ is real on the x axis, so reflects to an entire function. Differentiating one finds that F' is entire and is symmetric about the x-axis so is bounded everywhere since it is bounded in the upper half plane.

Liouville's Theorem implies that F' is constant. Denote that constant by a. Then $F - az$ has vanishing derivative so $F = az+b$ with constants a, b.

Since $\overline{F'} = \overline{a}$ must be parallel to the x-axis, a must be real. □

3. Flow in corners

Example 1.3 is a classic flow is defined in the positive quadrant $Q := \{(x,y)\ :\ x > 0, y > 0\}$ whose closure is denoted $\overline{Q} := \{x \geq 0\,,\, y \geq 0\}$. The complex potential is $F(z) = z^2$ has streamlines equal to the hyperbolas $xy =$ constant and the flow is tangent to the boundary curves that are the x and y axes. This gives a flow that turns through ninety degrees. However, the velocity $|F'(z)| = 2|z|$ diverges to infinity as $z \to \infty$ and one is tempted to reject the flow as unphysical and to search for a flow that turns the corner and has bounded velocities. No such flow exists.

Theorem 3.1. *Suppose that F is analytic in Q and continuously differentiable in $\overline{Q} \setminus 0$. Suppose in addition that the fluid velocity is tangent to the bounding curves and is uniformly bounded in Q. Then $F = \text{constant}$.*

Proof. On the boundary curve with $y = 0$ the velocity $\overline{F'}$ is tangent to the boundary so real. Therefore imaginary part ψ is constant on this boundary. Call that constant c. The function $F - ic$ has vanishing imaginary part on the real axis so by reflection extends to an analytic function in the right half space $x \geq 0$ with bounded velocity and flow tangent to $x = 0$. The theorem of the preceding section implies that $F = az + b$.

The corresponding velocity is therefore a constant vector field that is tangent to both the x and y axes. The constant vector must vanish so $F' = 0$ □

In support of the solution z^2 I offer two arguments. The fist and most telling is that the potential z^2 accurately describes flow near a corner as the example of flow over a semicircle shows. In that case the complex potential near the right angle corners has Taylor expansion $a(z - z_0)^2 +$ higher order terms. This shows that the quadratic potential gives a good approximation near the corner,

$$F' = 2a(z - z_0) + O(|z - z_0|^2)\,.$$

The second is that if F is a potential for a flow tangent to the boundaries of Q then the same is true of $F(\sigma z)$ for any $\sigma > 0$. The problem is dilation invariant. If would be natural to seek potentials that were invariant in the sense that for some function $c(\sigma)$ one had

$$F(\sigma z) \;=\; c(\sigma)\, F(z)\,.$$

In the homework, you showed that this implies that $F(z)$ is a homogeneous function of z. For Q that selects solutions z^{2n} with $1 \leq n \in \mathbb{N}$. The solution z^2 is the solution of slowest growth from this list.

Exercise 3.2. *Show that if $\theta = 2\pi/n$ with* **even** *integer $n \geq 2$, then an analogous reflection argument proves that there are no flows in the angular sector $0 < \arg z < \theta$ with bounded velocities tangent to the two sides.*

The case of $\theta = 2\pi/n$, for $n \geq 3$ and odd as well as angles that are irrational multiple of 2π require a different proof. Nevertheless, for those cases too, there are no flows with bounded velocities.

Exercise 3.3. *Suppose that $0 < \theta < \pi$. Show that there is no analytic F in the angular sector $S := \{0 < \arg z < \theta\}$ with F' uniformly bounded, continuous in $\overline{S} \setminus 0$ and with ∇u tangent to the bounding lines.* **Hint.** Use a conformal mapping to transport the flow from the sector to a half plane. Then use the half plane Theorem and pull back.

Exercise 3.4. *Suppose that F is analytic in the wedge $0 < \arg z < A$ where $A < \pi$ and the branch of* arg *takes values in $]-\pi, \pi[$. Suppose in addition that for any $\sigma > 0$,*

$$F(\sigma z) \;=\; c(\sigma)\, F\,, \tag{1}$$

for a suitable real $c(\sigma)$ depending on σ.

i. *Show that if σ and τ are two positive constants then $c(\sigma\tau) = c(\sigma)c(\tau)$.*

ii. *Show that c is a differentiable function on $]0,\infty[$.* **Hint.** *Consider a single fixed z.*

iii. *Show that F is a homogeneous function of z.* **Hint.** *Use real logarithms to nearly determine $c(\sigma)$.*

iv. *Show an analytic F on the wedge that satisfies (1) if and only if it is of the form $b\,z^\alpha$ for some real α and complex b.* **Hint.** *Read the earlier homework problem identifying analytic functions homogeneous of degree n with n integer.*

Exercise 3.5. *Continuation of the preceding problem. Find all flows in the wedge whose flow is parallel to the bounding lines and satisfies the symmetry (1) from the preceding assignment.* **Hint.** Use the result from that problem. **Discussion.** The flow velocity is bounded at the corner but the derivatives of the velocity are unbouded at the corner. The case of $A = \pi/4$ does not have this divergence.

Exercise 3.6. *Starting with flow in the unit disk swirling about the origin, find a swirling flow in the upper half disk so that the flow swirls about the point midway between the circle center and the circumference.* **Hint.** Map the domains. Then arrange that the point $0 + i/2$ goes to the center by performing an additional self map of the disk. The self maps of the disk were given in class.

Exercise 3.7. *Continuation. Show that near the corners of the half disk the flow resembles the flow with complex potential z^2 in a quadrant.* **Hint.** Find the leading term in the Taylor expansion of the complex potential at the corner. The potential of the transformed problem is the transform of the original potential.

Exercise 3.8. i. *For integer $n \geq 1$ show that the irrotational, incompressible, planar fluid flow with complex potential $F(z) = z^n$ is tangent to the boundary of the wedge $0 < \arg z < 2\pi/n$ and each of its the wedges obtained by rotating by $k2\pi/n$ with $k \in \mathbb{Z}$.*

ii. *Sketch the streamlines.* **Hint.** The streamlines satisfy $\operatorname{Im} z^n = c$. So z belongs to the image of a $\{\operatorname{Im} w = c\}$ by the appropriate branch of $z = w^{1/n}$.

iii. *When n is even, show that the flow is tangent to the boundary of $\{y > 0\}$ and also to the boundary of the positive quadrant Q.*

4. No flows in bounded Jordan domains

The swirling flow equal to $\nabla\theta$ has streamlines that are circles with center at the origin. This yields flows on the annular regions $r < |z| < R$. It is rather remarkable that no such flows can exist on the domains that are bounded by nice Jordan curves.

Theorem 4.1. *Suppose that Ω is a bounded open set that is the interior of a continuously differentiable Jordan curve. Then the only continuously differenentiable incompressible, irrotational flows on Ω whose velocity is tangent to the boundary are those with velocity equal to zero.*

Proof. Write $F = \phi + i\psi$. The velocity is tangent to the boundary so the boundary is either a single streamline or a family of streamlines separated by stagnation points. Since ψ is constant on each streamline it follows from the continuity of ψ that ψ is constant on the boundary. Then ψ is harmonic in Ω and constant on $\partial\Omega$ so by uniqueness for the Dirichlet problem, ψ is constant in Ω. The Cauchy-Riemann equations imply that $\nabla\phi = 0$ and therefore that ϕ is also constant. $\square$

This theorem shows that the swirl in Exercise 3.6 is essential. The corresponding flows have curl equal to a Dirac delta at the center of the swirl. Theorem 4.1 demands that the curl vanishes throughout Ω.