Image of Spheres by Invertible M

Abstract This note presents a solution of the following problem which avoids conformal matrices and polar decomposition.

Problem. Show that the image of a sphere by an invertible linear transformation on $\mathbb{R}^N$ is an ellipsoid. Find an algorithm to compute the lengths and directions of the principal axes.

1 Scalar product.

Denote the *standard scalar product* of vectors in $\mathbb{R}^n$ by

$$\langle x\,,\,y\rangle \;=\; \sum x_i\,y_i\,.$$

Vectors x and $\widetilde{x}$ are **orthogonal** if and only if $\langle x\,,\,\widetilde{x}\rangle = 0$. An **orthonormal basis** for $\mathbb{R}^n$ is a set of mutually orthogonal unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$.

2 Transposes.

Suppose that A_{ij} is an $n \times n$ real matrix. The **transpose** A^t of A is defined by

$$(A^t)_{ij} \;:=\; A_{ji}\,.$$

The matrix of the transpose is the matrix of A flipped in the diagonal.

Example 2.1.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t \;=\; \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\,.$$

Proposition 2.2. *For all vectors x and y, $n \times n$ matrices A and B, and real numbers α,*

i. $(A + B)^t = A^t + B^t$,

ii. $(\alpha A)^t = \alpha A^t$,

iii. $(AB)^t = B^t A^t$,

iv. $\langle Ax\,,\,y\rangle \;=\; \langle x\,,\,A^t y\rangle$.

3 Positive symmetric matrices.

Definition 3.1. *A* **symmetric real matrix** *R is one for which $R_{ij} = R_{ji}$ for all i, j. Equivalently $R = R^t$.*

It is a fundamental fact that for every symmetric matrix there is an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for $\mathbb{R}^n$ consisting of eigenvectors of R,

$$R\,\mathbf{e}_j = \lambda_j \mathbf{e}_j, \qquad j = 1, \ldots n\,.$$

Expressing an arbitrary vector x in this basis yields

$$x = \sum \alpha_j \mathbf{e}_j, \qquad \alpha_j = \langle x, \mathbf{e}_j \rangle.$$

Then,

$$Rx = \sum \alpha_j R\mathbf{e}_j = \sum \alpha_j\, \lambda_j\, \mathbf{e}_j\,.$$

The coordinate α_j in the basis $\mathbf{e}_j$ is multiplied by λ_j.

Definition 3.2. *A symmetric real R is* **positive** *when all the eignevalues are strictly positive.*

Proposition 3.3. *A symmetric matrix R is positive if and only if for all $x \neq 0$, $\langle Rx, x\rangle > 0$.*

Proof. Compute in an orthonormal basis of eigenvectors,

$$Rx = R(\sum \alpha_j \mathbf{e}_j) = \sum \alpha_j R\mathbf{e}_j = \sum \alpha_j \lambda_j \mathbf{e}_j.$$

Then compute

$$\langle Rx, x\rangle = \Big\langle \sum \alpha_j \lambda_j \mathbf{e}_j\,,\, \sum \alpha_k \mathbf{e}_k \Big\rangle = \sum \alpha_j \alpha_k \lambda_j \langle \mathbf{e}_j, \mathbf{e}_k\rangle\,. = \sum \lambda_j \alpha_j^2$$

The proposition follows. ☐

Proposition 3.4. *If M is invertible then $M\,M^t$ is a positive symmetric matrix.*

Proof. Compute,

$$(MM^t)^t \;=\; (M^t)^t M^t \;=\; MM^t,$$

proving symmetry.

For $x \neq 0$,

$$\langle M\,M^t x\,,\, x\rangle \;=\; \langle M^t x\,,\, M^t x\rangle \;=\; \|M^t x\|^2 \;>\; 0\,,$$

proving positivity. ☐

4 Image of a ball by an invertible M.

Theorem 4.1. *If M is an invertible linear transformation denote by $\mathbf{e}_j$ and $\lambda_j > 0$ an orthonormal basis of eigenvectors of $M\,M^t$ and the corresponding eigenvalues. Then, the image of the unit sphere $\{\|x\| = 1\}$ by M is the ellipsoid with principal axes of length $2/\sqrt{\lambda_j}$ along the directions $\mathbf{e}_j$.*

Proof. The image of the unit sphere is the set of vectors $y = Mx$ with $\|x\| = 1$. Write $x = M^{-1}y$. The y are characterized by the equation $\|M^{-1}y\|^2 = 1$. Compute

$$\|M^{-1}y\|^2 \;=\; \Big\langle M^{-1}y\,,\, M^{-1}y\Big\rangle \;=\; \Big\langle (M^{-1})^t M^{-1}y\,,\, y\Big\rangle\,.$$

One has,

$$(M^{-1})^t M^{-1} \;=\; (MM^t)^{-1}\,.$$

Therefore, the vectors $\mathbf{e}_j$ are eigenvectors of $(M^{-1})^t M^{-1}$ with eigenvalue λ_j^{-1}. In the basis $\mathbf{e}_j$ with $y = \sum \beta_j \mathbf{e}_j$ the image of the unit sphere is given by

$$\|M^{-1}y\|^2 \;=\; \sum \frac{\beta_j^2}{\lambda_j} \;=\; 1, \qquad \sum \beta_j^2 = 1\,.$$

In the coordinates β_j one has (by definition of ellipsoid) an ellipsoid with axes along the coordinate axes and of length $2/\sqrt{\lambda_j}$.

Since the coordinate axes in the β coordinates are the directions $\mathbf{e}_j$ in the original coordinates, this completes the proof. □

Example. *Compute the image of the unit circle by the Jacobian matrix*

$$J \;=\; \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

from class.

Solution. Compute

$$J\,J^t \;=\; \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \;=\; \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\,.$$

The eigenvalues are the roots λ of,

$$0 = \det \begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} \;=\; (5-\lambda)^2 - 9 = \big((5-\lambda)+3\big)\,(5-\lambda)-3\big)\,.$$

The roots are $\lambda = 5 \pm 3$. Eigenvalues and unit eigenvectors are

$$\lambda_1 = 2, \quad \mathbf{e}_1 = (1/\sqrt{2}\,,\, -1/\sqrt{2}), \qquad \lambda_2 = 8, \quad \mathbf{e}_2 = (1/\sqrt{2}\,,\, 1/\sqrt{2})\,.$$

The image is the ellipse with major axis along the direction $(1, -1)$ with length $2/\sqrt{2}$ and minor axis along the direction $(1, 1)$ and length $2/\sqrt{8}$.