

## Neumann Problems, Insulator Boundary Condition

**Summary.** *If one knows a conformal map of a domain  $G$  to the upper half space, then steady state temperatures can be computed when the boundary consists of two or three intervals exactly one of which is insulated and each of the others is at constant temperature.*

### 1. INTRODUCTION

This note uses complex variables to solve by explicit formulas some mixed boundary value problems for harmonic functions. Harmonic functions represent steady state temperature distributions. The heat current is given by

$$J := -\kappa \operatorname{grad} u$$

with conductivity  $\kappa$  assumed to be constant.

Heat flow is studied in a domain  $G$ . On parts of the boundary, the temperature is assigned. In practice this corresponds to being in contact with a heat or cooling device. On intervals where the assigned steady state temperature is constant this can be maintained by contact with a large constant temperature bath.

Part of the boundary is assumed to be insulated. On such parts the heat current has vanishing component in the direction of the normal vector. This hold if and only if  $\mathbf{n} \cdot \operatorname{grad} u = 0$ . Equivalently the normal derivative satisfies the Neumann boundary condition

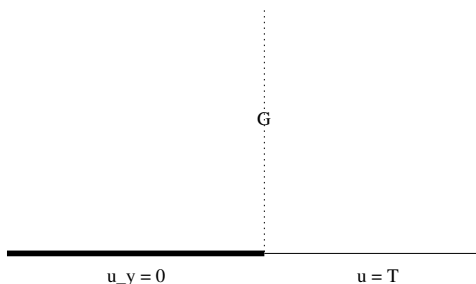
$$\frac{\partial u}{\partial n} = 0.$$

### 2. TWO BOUNDARY INTERVALS

The first class of problems that we discuss have part of the boundary at one fixed temperature  $T$  and the rest of the boundary insulated. An example is the freezer compartment of a refrigerator. Like that example, it is reasonable to think that over a long period of time, the temperature  $T$  will invade the entire region and the steady state solution will be at constant temperature  $T$ , The constant function  $u = T$  is harmonic, assumes the boundary value  $T$  at the fixed temperature parts of the boundary, and has vanishing normal derivative at the insulated parts. It is a solution of boundary value problem that is expected to describe the steady state. The mathematical difficulty is to show

that this is the only bounded solution. A beautiful application of the methods of complex analysis proves uniqueness in pleasingly rich set of circumstances.

The first example is heat flow in the upper half space  $G := \{\text{Im } z > 0\}$ . The boundary segment on the positive real axis is held at temperature  $T$ . The boundary segment on the negative real axis insulated. The insulated segment is indicated by a thick line in the figure.



**Theorem 2.1.** *The only bounded harmonic function in  $G$  that is continuous up to the strictly positive real axis and continuous differentiable up to the strictly negative real axis and satisfies the boundary conditions in the figure is the function  $u = T$ .*

*Proof.* Only uniqueness needs to be proved. If there were two solutions  $u_1$  and  $u_2$  then the difference  $u := u_1 - u_2$  is a solution with  $T = 0$ . It suffices to show that the only bounded solution with  $T = 0$  is  $u = 0$ .

Since the domain  $G$  is simply connected, there is a harmonic conjugate  $v$  so  $F = u + iv$  is analytic in  $G$ . The Cauchy-Riemann equation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (2.1)$$

shows that along the negative real axis,  $v_x = 0$  so  $v$  is constant on the negative real axis. Adding a constant to  $v$  one has  $v = 0$  on the negative real axis. Then  $F$  is real valued on  $\mathbb{R} \setminus 0$ .

The Schwartz Reflection Principle implies that the function equal to  $F$  in the upper half plane and  $\overline{F(\bar{z})}$  in the lower half plane is analytic in  $\mathbb{C} \setminus 0$ . Slightly abusing notation we denote this analytically continued function as  $F$ .

The function  $F$  has bounded real part and an isolated singularity at 0. It follows that the singularity is removable and  $F$  extends to an entire function, still denoted  $F$ . The extended function is entire with bounded real part so Liouville's theorem, as in the proof of uniqueness of the Dirichlet problem, implies that  $F$  is constant.

Therefore,  $u = \operatorname{Re} F$  is constant. Since  $u = 0$  on the positive real axis, it follows that  $u = 0$  everywhere.  $\square$

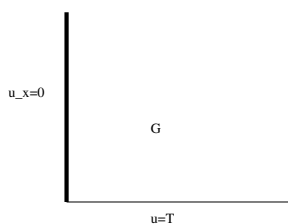
### 3. CONFORMAL TRANSFORMATION OF THE NEUMANN CONDITION

For the Dirichlet problem, one maps one domain to another by a conformal mapping, the Dirichlet boundary conditions in one domain are transported to corresponding Dirichlet boundary conditions in the transformed domain. In this way solving in a half space for piecewise constant data transfers to solutions in any domain that can be mapped to the half space.

The same is true for boundary value problems involving insulators. The important observation is that if one can solve on one domain  $G_1$  with insulator section  $\Gamma_1 \subset \partial G$  and there is conformal mapping to  $\zeta = f(z)$  from  $G_1 \rightarrow G_2$  taking  $\Gamma_1 \rightarrow \Gamma_2 \subset \partial G$ , then  $u(\zeta)$  satisfies the insulator boundary condition on  $\Gamma_2$  if and only if  $u(\zeta(z))$  satisfies the insulator boundary condition on  $\Gamma_1$ . This is so because  $f$  maps the boundary direction to  $G_1$  to the boundary direction on  $G_2$ . Conformality implies that it maps the normal direction to normal direction also. Therefore normal derivative equal to zero is transported by the mapping.

In this way the preceding Theorem shows that on essentially arbitrary simply connected domains, the unique steady temperature with one insulator and one Dirichlet temperature  $T$  is the constant function  $u = T$ .

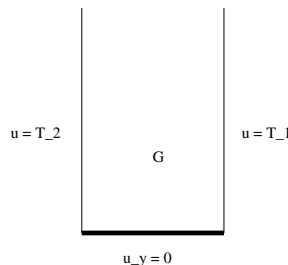
**Example 3.1.** *The map  $\zeta = z^2$  transforms the problem*



*to the problem in the earlier figure.*

### 4. THE U SHAPED DOMAIN INSULATED AT THE BOTTOM

The point of departure is the problem summarized by the figure



This problem is solved by inspection in science courses as follows. Consider the steady temperature distribution between two vertical lines at different constant temperatures. The temperature  $u = ax + b$  is linear in  $x$  and the heat current is horizontal.

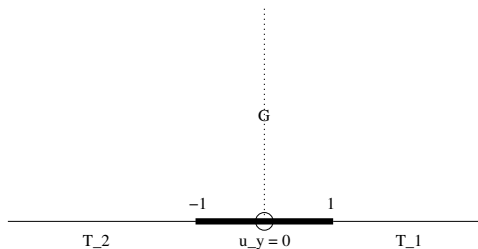
Therefore if one inserts a horizontal boundary between the lines to make a U-shaped region, the heat flow is tangent to the new boundary segment. Therefore the Neumann boundary condition is satisfied on the horizontal boundary. In this way the boundary value problem of the figure is solved by a harmonic function  $u = ax + b$ .

This solution can be found by purely complex methods as follows. Denote by  $u$  a bounded harmonic solution. Since  $G$  is simply connected there is a harmonic conjugate  $v$  and associated analytic  $F = u + iv$ . The Cauchy-Riemann equation (2.1) implies that  $v$  is constant on the insulated boundary. Adding a constant to  $v$  one may assume that  $v = 0$  on this boundary. Then  $F$  is real at this boundary segment. Schwartz reflection yields an analytic function, still denoted  $F$ , on the whole strip between the two vertical lines. The real part of that  $F$  is bounded and harmonic on the region between the lines and equal to  $T_j$  on the  $j^{\text{th}}$  boundary line. It follows from the study of the Dirichlet Problem that  $u = ax + b$ .

## 5. THE $\sin z$ MAP

The map  $e^z$  maps the half strip  $\{0 < y < \pi, -\infty < x < 0\}$  conformally to the interior of the upper half of the unit disk. Then a linear fractional transformation maps to the upper half space. In this way one can find a mapping from the U domain in  $\{0 < y, -\pi/2 < x < \pi/2\}$  to the upper half space and the insulated region on the  $x$ -axis to  $] -1, 1[$ . That mapping is equal to  $\sin z$ .

This mapping allows one to solve the problem indicated in the next figure.



The bounded harmonic function  $u$  solves this problem if and only if  $u(\sin z)$  is the unique bounded solution of the problem in the U-domain in  $-\pi/2 < x < \pi/2$ .

**Theorem 5.1.** *For the upper half space problem indicated by the above figure with insulator in  $-1 < x < 1$  there is unique bounded harmonic solution that extends smoothly to the boundary with the exception of the points at the edge of the insulated segments. The solution is uniquely determined by*

$$u(\sin z) = ax + b, \quad a(-\pi/2) + b = T_1, \quad a(\pi/2) + b = T_2.$$

By translation in  $x$  and dilation in  $x, y$  one can solve the analogous upper half space problem where the insulated interval is arbitrary. Therefore if  $G$  is any simply connected domain easily mapped to the halfspace and the boundary is split into three intervals by two boundary points  $P_1$  and  $P_2$  then the problem with temperatures on two of the intervals and the third insulated is explicitly solvable. And there are uniqueness results that go with it.

**Example 5.2.** *In the U-domain one can solve the two temperatures separated by an interval of insulator in arbitrary position. For example on the left line or on the left line touching the bottom. Or overlapping one of the corners. Or a subset of the bottom segment, .... etc.*

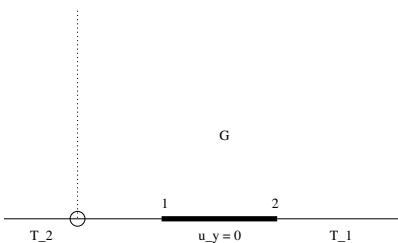
**Example 5.3.** *For  $G$  equal to the positive quadrant the mapping  $z^2$  allows one to solve the two temperatures separated by an interval of insulator with insulator arbitrarily placed on the boundary.*

**Example 5.4.** *If  $G$  is the unit disk one can solve the two temperature and one insulator problem for arbitrary interval of insulator on the boundary.*

**Example 5.5.** *An example that is **not** treated this way is the boundary value problem in the half space with an insulator interval and in the complement the temperature is equal to zero except for a small interval of width  $\varepsilon$  where the temperature is  $1/\varepsilon$ . If one could solve, then passing to the limit would give the Greens' function and thereby solving for*

general boundary temperatures. However, on the  $x$ -axis there is only the point at infinity where the temperature changes while the present problem has two finite points with temperature change. No mapping can transform one to the other.

**Exercise 5.6.** Use the  $\sin z$  map to solve the problem sketched below.



**Exercise 5.7.** A domain occupies the upper half of the unit disk and is insulated along the horizontal diameter. The left hand of the top is at temperature  $-1$  and the right hand side of the top at temperature  $1$ . Find the steady state temperature distribution. **Hint.** The solution is an odd function of  $x$  and your solution should clearly show this.