Orientation

1 Orientation on $\mathbb{R}^2$.

Definition 1.1 *In the plane $\mathbb{R}^2$ an orientation defines which direction of rotation is the positive sense. An orientation is prescribed by giving an* **ordered** *pair of linearly independent vectors*

$$(v, w), \qquad v = (v_1, v_2), \quad w = (w_1, w_2).$$

By definition the direction of rotation from v to w through an angle less than π is the positive direction.

Example 1.1 *The standard orientation on $\mathbb{R}^2$ and therefore on $\mathbb{C}$ is given by the ordered pair*

$$(1, 0), \qquad (0, 1).$$

In the standard graphical representation of $(x, y) \in \mathbb{R}^2$ with x increasing to the right and y increasing upwards the positive sense of rotation is counterclockwise.

The ordered linearly independent pair (v, w) defines the same orientation if and only if

$$\det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} > 0.$$

Negative determinant pairs define the opposite orientation.

2 Positive sense on a Jordan curve.

Definition 2.1 *A differentiable Jordan curve is a continuously differentiable closed curve without self intersections.*

Such a curve is also called a simple closed curve. The Jordan Curve Theorem asserts that in the plane the complement of a Jordan curve has exactly two open connected components, one of which is bounded. The bounded component is called the **interior**. Though this result is difficult for continuous Jordan curves, the differentiability permits a fairly straight forward proof using the Inverse Function Theorem (see the Differential Topology text by Guillemin and Sternberg).
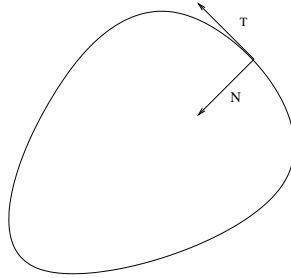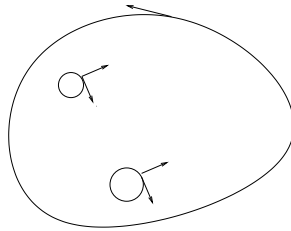
Figure 1: Positive sense on a Jordan curve.

Figure 2: Positive sense of region with holes.

The orientation on $\mathbb{R}^2$ determines a positive sense, equivalently a direction of motion, on a Jordan curve. It is the counterclockwise sense. Equivalently, the ordered pair consisting of the unit forward tangent vector and the inward unit normal form a positively oriented pair in $\mathbb{R}^2$. An intuitive description is that an airplane moving tangent to the boundary with its nose in the positive sense will have its *left* wing pointing into the domain.

3 Positive sense on a more complicated boundary.

The prescription, either with airplane with left wing inward, or unit forward tangent and interior unit normal being positively oriented tells which way to orient the boundary of holes in the interior of a domain. In the figure the domain is the inside of a Jordan curve with two disks punched out. This creates two circular boundaries at the edges of the disks. The positive sense on these interior circles is counterclockwise.

Exercise 3.1 *For the region $R := \{z : |z| > 1\}$ find the positive sense*

on the boundary using both the airplane and the unit tangent, unit interior normal recipes.

Exercise 3.2 *A hiker is walking in the plane starting in the exterior of a farm bounded by a Jordan curve traced by a fence. The hiker crosses the fence. He continues walking and finds the the farm has a VERY complicated shape. After crossing the fence 1019 times over the course of a week, is he inside or outside the farm?*

Discussion. This is one of the two key ideas in proof of the Jordan Curve Theorem in Guillemin and Pollack. The second idea is that of a collar neighborhood of the boundary.