Image of Circles by Invertible 2×2 Matrices

1 Characterization of circle preserving matrices

Theorem 1.1. *The image of the unit circle in $\mathbb{R}^2$ by an invertible 2×2 matrix is an ellipse with center at the origin. It is a circle if and only if the rows of the matrix are orthogonal vectors of equal length.*

Proof. Denote the invertible matrix by

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \neq 0\,. \tag{1.1}$$

Then

$$M^{-1} \;=\; \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

M maps the unit circle to

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} \;=\; M \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x^2 + y^2 = 1 \right\}.$$

The equation of this set in u, v space is

$$\left\| M^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 \;=\; 1\,.$$

Plugging in the formula for M^{-1} yields the equation

$$\frac{1}{(ad-bc)^2} \left\| \Big(du - bv\,,\, -cu + av \Big) \right\|^2 \;=\; 1\,.$$

Expanding yields

$$\frac{1}{(ad-bc)^2} \Big((du - bv)^2 \;+\; (-cu + av)^2 \Big) \;=\; 1\,.$$

Simplify to find

$$(d^2 + c^2)u^2 \;-\; 2(ac + bd)uv \;+\; (a^2 + b^2)v^2 \;=\; (ad - bc)^2\,.$$

This quadratic equation describes the image of the unit circle so is a bounded closed curve. As a quadric curve it is a conic section or a degenerate section.

That is an ellipse, hyperbola, parabola or degenerate form, a line, two lines, a point, or empty. Since the set contains an infinite number of points (M is one to one and the circle is infinite) and is compact (M is continuous and the circle is compact) the only candidate is an ellipse.

If (u, v) is a solution, then so is $(-u, -v)$ so the center of the ellipse must be the origin.

Write the equation as

$$Au^2 + Buv + Cv^2 \;=\; D > 0\,. \tag{1.2}$$

This describes a circle centered at the origin exactly when $A = C > 0$ and $B = 0$. Since A and C are the lengths of the rows of M and B is the scalar product of the rows this proves that the condition announced in the Theorem is necessary and sufficient. □

Remark. The condition on the rows is equivalent to the analogous condition that the columns of the matrix are orthogonal vectors of equal length. One condition says that MM^t is a constant multiple of the identity. The other says that M^tM has this property. Each condition follows from the other on taking transpose.

Example. For the Jacobian matrix from class,

$$J \;=\; \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix},$$

the rows are of equal length but not orthogonal. The columns are orthogonal but are not of equal length. Either way one sees that the image of circles are ellipses that are not circles.

A typical Jacobian will neither have rows of equal length nor rows that are orthogonal. Most of the time, Jacobians map circles to ellipses that are not circles.

2 Matrix of multiplication by a complex number

Theorem 2.1. *If M is linear and invertible from $\mathbb{R}^2$ to itself then the following are equivalent.*

1. *M maps circles to circles and has positive determinant.*

2. *There is a nonzero complex number w so that the the image by M of $x + iy$ is equal to $w(x + iy)$.*

Proof. **2** $\Rightarrow$ **1**. Write $w = \alpha + i\beta$ with real α, β. Expanding

$$w(x+iy) \;=\; (\alpha+i\beta)(x+iy) \;=\; (\alpha x - \beta y) + i(\beta x + \alpha y)$$

shows that the matrix of the linear transformation $x+iy \mapsto (\alpha+i\beta)(x+iy)$ is equal to

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

The columns are orthogonal and of equal length so circles are mapped to circles. The determinant is $\alpha^2 + \beta^2 > 0$ completing the proof of this implication.

1 $\Rightarrow$ **2**. Denote the entries of the Matrix M as in equation (1.1). Since the second column of M must be orthogonal to the first column and of the same length one must have

$$(c,d) \;=\; \pm(-b,a)\,.$$

The determinant is equal to $\pm(a^2+b^2)$. Since the determinant is positive one must have

$$M \;=\; \begin{pmatrix} a & -c \\ c & a \end{pmatrix}.$$

This is the matrix of multiplication by $w = a+ic$ proving the implication. □