Electrostatic Screening

1. Electrical screening I, surrounded bounded domain

In a region free of charge, the electrostatic potential $V(x)$ is a harmonic function. If a bounded region $\mathcal{R}$ of space is surrounded by boundaries kept at the same potential, the potential satisfies

$$\Delta V \;=\; 0 \quad \text{in} \quad \mathcal{R}, \qquad V\big|_{\partial\mathcal{R}} \;=\; C\,. \tag{1.1}$$

Theorem 1.1. *In these cases static electric field vanishes in the enclosed domain.*

Remark 1.2. *The presences of static electric charges and currents outside cannot create an electric field on the inside. It is perfectly shielded.*

Proof. The function $V = C$ is one solution of (1.1). Uniqueness of solutions follows from the maximum and minimum principals for harmonic functions. The field equal to $\operatorname{grad} V$ vanishes. ☐

If the domain $\mathcal{R}$ is the annular region $r_1 \le |z| \le r_2$, the unique harmonic function on $\mathcal{R}$ that has constant potential V_1 on the inner boundary and V_2 on the outer boundary is of the form $A \ln r + B$.

Exercise 1.3. *Find A and B.*

This is an example with two boundary components that are not kept at the same potential.

2. Electrical screening II, surrounded strip

Consider the region $-\infty < x_l < x < x_r < \infty$ bounded on the left and right by perfectly conducting planes that are both at the same constant potential V. The reflection argument of the Fluid Flow handout shows that the unique bounded harmonic function in this unbounded region that assumes the given boundary values is constant throughout the domain between the plates. One has an unbounded region shielded in the sense of vanishing electric field, from the effects of static external electrical sources.

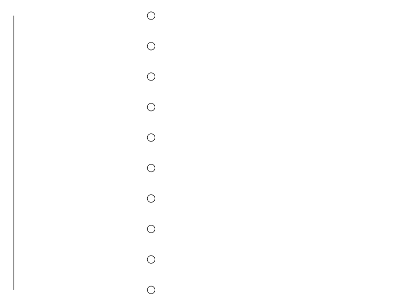

FIGURE 1. $V = V_l$ on left and screen, $V = V_r$ on right.

3. ELECTRICAL SCREENING III, THE FARADAY CAGE

In practice one screens from external electrical influence by encircling the domain to be protected by a screen of electrical wires. Such a screen is called a *Faraday cage*. There are two important dimensions, the spacing ℓ between adjacent wires and the radii r of the wires. The interplay of those parameters in estimating the effectiveness of the screen is a difficult problem that is discussed using complex functions in Articles 203-205 of volume I of Maxwell's *Treatise on Electricity and Magnetism.* Analytic functions yield harmonic functions that serve to construct an approximate solution that reveals the important qualitative behaviors.

Consider first the slab $\{x_l < 0 < x_r\}$. If the left hand boundary is a conductor at potential V_l and the right hand a second conductor at potential V_2 then the the unique bounded harmonic function in the slab attaining these boundary values is

$$V_l \; + \; \frac{V_r - V_l}{x_r - x_l} \, (x - x_l) \, .$$

The electrical field is constant in the slab.

The problem addressed in this section is to find the electric field when a screen of parallel wires with $r < \ell << \min\{|x_l|, x_r\}$ is placed in the plane $x = 0$ and kept at the same potential V_l as the left hand conducting plate (see Figure 1). The goal is to estimate the effectiveness of the electrical shielding as a function of the parameters r, ℓ, x_l, x_r.

3.1. **The three harmonic building blocks.** The strategy is to find an approximate solution as a linear combination of three harmonic functions ϕ_j so that each ϕ_j has nearly constant values on each of the three bounding sets, the left and right hand planes, and the screen. Two of the functions are the constant function 1 and the function x.

The linear function x is nearly constant on the screen since $\rho << 1$. The brilliant idea of Maxwell is to construct a third function as follows. The function $\ln r$ is harmonic on $\mathbb{R}^2 \setminus 0$ and its level sets $\{\ln r = c\}$ for $c << -1$ are small circles about the origin. Suppose next that $F(z)$ is analytic with $F(\underline{z}) = 0$ and $F'(\underline{z}) \neq 0$. Then for $z \approx \underline{z}$,

$$\begin{aligned} F(z) &\approx F(\underline{z}) + F'(\underline{z})\,(z-\underline{z}) \;=\; F'(\underline{z})\,(z-\underline{z}), \\ |F(z)| &\approx |F'(\underline{z})|\,|z-\underline{z}|\,. \end{aligned}$$

Therefore the level set

$$\Big\{z \;:\; \mathrm{Re}\big(\ln(F(z)) = \ln(|F(z)|) = c << -1\Big\}$$

contains a closed curve very close to a small circle centered at $\underline{z}$.

This is applied to the function $F(z) = e^{2\pi z/\ell} - 1$ that is analytic and periodic with period $i\ell$. It vanishes at the points $i\ell n$ with $n \in \mathbb{Z}$ and its derivative at those points is equal to $(2\pi/\ell)$. Therefore on the boundary of the cylinders of radius ρ centered at $i\ell$ one has

$$\ln|F(z)| \;\approx\; \ln\left(\frac{2\pi\rho}{\ell}\right).$$

This shows that $\ln|F(z)|$ is nearly constant on the screen.

Next examine $\ln|F(z)|$ on the planes $x = x_l$ and $x = x_r$. Since $x_l << -\ell$, $e^{2\pi z/\ell} \approx 0$ when $x = x_\ell$ so $|F(z)| \approx 1$ and therefore $\ln|F(z)| \approx 0$. Since $x_r >> \ell$ on $x = x_r$ one has

$$\ln|F(z)| \;\approx\; \ln e^{2\pi x_r/\ell} \;\approx\; \frac{2\pi x_r}{\ell}\,.$$

In particular *the harmonic function* $\ln|F(z)|$ *is nearly constant on each of the left and right hand bounding planes and also on the screen.*

3.2. **The approximate solution.** Seek an approximate solution

$$\phi \;:=\; A\,\ln|F(z)| \;+\; B\,x \;+\; C\,.$$

Using the approximate values of the three building blocks on the three conductors the boundary conditions at $x = x_\ell$, the screen, and $x = x_r$ yield the three equations

$$\begin{aligned} A\cdot 0 \;+\; B\,x_l \;+\; C &= V_1\,, \\ A\,\ln\frac{2\pi\rho}{\ell} \;+\; B\cdot 0 \;+\; C &= V_1\,, \\ A\,\frac{2\pi x_r}{\ell} \;+\; B\,x_r \;+\; C &= V_2\,. \end{aligned}$$

Example. An interesting limit is $\rho \to 0$ with all the other parameters fixed. In this case the middle equation implies that $A \to 0$ so the potential and field approach those of the problem without the screen. This physically reasonable conclusion is not a consequence of the standard physics text explanation of screening.

Return to the general case. The middle equation is used to eliminate C,

$$C \;=\; V_1 - \kappa\, A\,, \qquad \kappa := \ln \frac{2\pi\rho}{\ell}\;.$$

The first equation yields

$$x_l\, B \;=\; V_1 - C \;=\; \kappa\, A\,.$$

The third equation then yields

$$A\,\frac{2\pi x_r}{\ell} \;+\; A\,\frac{\kappa x_r}{x_l} \;+\; (V_1 - \kappa\, A) \;=\; V_2\,,$$

$$A\Big(\frac{2\pi x_r}{\ell} \;+\; \frac{\kappa\, x_r}{x_l} \;-\; \kappa\Big) \;=\; V_2 - V_1\,.$$

Example. In typical experimental situations the coefficient of A in this equation is dominated by the first summand and one finds

$$A \;\approx\; \frac{\ell(V_2 - V_1)}{2\pi x_r}\,, \qquad B \;\approx\; \frac{\kappa\,\ell(V_2 - V_1)}{2\pi x_\ell\, x_r}\,.$$

In units so that x_l and x_r are order one, $\rho < \ell << 1$. The coefficient B in the absence of screen would be $(V_2 - V_1)/(x_r - x_l)$. The new B is $\kappa\ell/2\pi$ times a quantity of the same magnitude. Since the field in the domain between the screen and $x = x_l$ is approximately equal to B, the screening efficiency is measure by the factor

$$\kappa\,\ell/2\pi \;=\; (\ell/2\pi)\;\ln\left(\frac{\ell/2\pi}{\rho}\right).$$

If ρ/ℓ is kept fixed then the field in the screened area has order of magnitude ℓ times the field in the absence of the screen. The reader is warned that this approximation is not accurate when $\rho << \ell$ in which case neglecting the κ term in the equation for A is not justified.

Using the maximum and minimum principals for harmonic functions it is not hard to give rigorous error estimates for the approximations above.