Math 555 Final Exam, December 20, 2002, Prof. J. Rauch Name ....

Instructions. **1.** Two 3in. × 5in. cards of notes from home are allowed. Closed book.
2. Show work and explain clearly.
3. 100 points total
4. There are 7 questions and 9 pages. The first question is has 7 short parts.

1. (35 = 7 × 5 points) **a.** Evaluate exactly

$$\oint_{|z|=5} \frac{e^{\pi z}}{z-i}\, dz\,.$$

b. Find the Laurent expansion which represents the function

$$g(z) \;=\; \frac{1}{z-1} + \frac{1}{z-2}$$

in the annulus $1 < |z| < 2$.

Prob.	1	2	3	4	5	6	7
Total	35	10	10	13	10	12	10
Score							

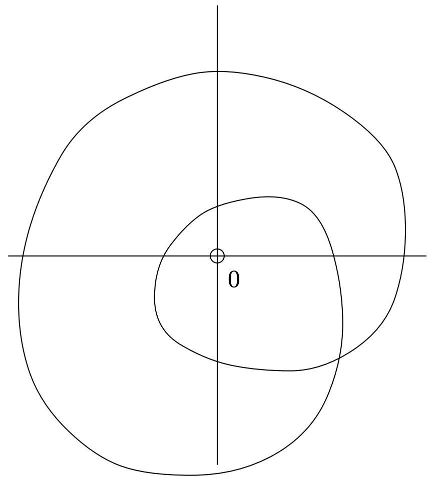
In each of the next three questions you must determine whether the given function has a zero at $z = 0$ and if so what order, or a pole at $z = 0$ and if so what order, or an essential singularity at $z = 0$ in which case you must compute the residue.

c. $\left(e^{1/z}\right)^2$.

d. $\left(e^{-z} - 1\right)^2$.

e. $\left(e^{z} - 1\right)^{-2}$.

The function $f(z)$ is analytic on the closed unit disk $\{z : |z| \le 1\}$. It maps the boundary of the disk one to one onto the curve in the figure



f. How many zeroes (counting multiplicity) does f have inside the disk? Explain.

g. Indicate with an arrow on the figure the direction in which the curve is traversed when z goes around the unit circle in the counterclockwise (= positive) direction.

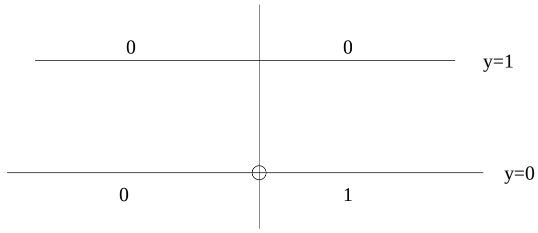
2. (10 points) Compute all the possible harmonic conjugates on $\mathbb{C}$ of the function $u(x, y) = x - y$.

3. (5+5 points) The principal branch $\operatorname{Log} z$ of the logarithm is defined in $\mathbb{C}\backslash]-\infty, 0]$ by $-\pi < \operatorname{Arg} z < \pi$. The branch cut is along the negative real axis.

a. Find the radius of convergence of the Taylor series of $\operatorname{Log} z$ centered at 1. Explain.

b. Find the radius of convergence of the Taylor series of $\operatorname{Log} z$ centered at $i-1$. Explain.

4. (10+3 points) **a.** Find the unique bounded steady state temperature u in the strip $0 < y < 1$ which assumes the boundary values indicated in the figure.



b. Heat flows into the strip along the positive real axis and out along all the other boundaries. Suppose that the heat conductivity is equal to 1 so that the heat current is exactly equal to $-\operatorname{grad} u$. Compute the rate at which heat flows into the strip along the segment $0 < x < 1$ of the positive x axis.

5. (10 points) Compute the exact value of

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\, dx\,.$$

6. (2+10 points) **a.** Explain why the following integral can only be interpreted as a principal value.

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x(x^2+1)}\, dx\,, \qquad \xi \in \mathbb{R}.$$

b. For $\xi \in \mathbb{R}$ compute the exact value of

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x(x^2+1)}\, dx\,.$$

7. (4+2+4 points) The analytic function $F(z) = z^2$ is the complex potential of an incompressible, inviscid, irrotational fluid flow in the first quadrant $x > 0, y > 0$. The streamlines are the hyperbolas $xy = \text{const.} > 0$ which describe a flow which "turns the corner" at the origin.

a. Show that as $z \to \infty$ in the first quadrant, the speed of the fluid flow defined by F tends to infinity.

The observation in **a.** suggests that the flow is not relevant in real world situations. In fact, it is surprisingly relevant as the next computations reveal.

The function $G(z) = z + 1/z$ is the complex velocity potential for steady flow in the part of the upper half plane $y > 0$ which is outside the the cylinder $|z| \leq 1$. The flow velocity tends to (1,0) as $z \to \infty$.

b. Find the Taylor series of G up to and including quadratic terms at the base point $z = 1$.

c. Subtracting a constant from the complex potential does not change the flow. Similarly multiplying by a constant does not change the streamlines, only the speed at which they are traveled. Find the scaling factor $C(\epsilon)$ so that the zooms of G at $z = 1$ defined by

$$G^\epsilon(w) := C(\epsilon)\,\big(G(1 + \epsilon w) - G(1)\big)$$

converge as $\epsilon \to 0$ to $F(w)$ for every w in the positive quadrant.

Discussion. 1. The convergence is uniform on bounded subsets of w. **2.** The seemingly unphysical flow defined by F yields a good approximation near $z = 1$ for the flow defined by G. Close to the corner the streamlines are "infinitesimal hyperbolas".