The Laplace Transform and Partial Fractions

Suppose that $f(t) : [0,\infty[\to \mathbb{C}$, grows no faster than exponentially, that is there are real constants M and A so that,

$$|f(t)| \ \leq \ M\, e^{At}\,.$$

The Laplace transform of f is then defined for $\mathrm{Re}\, s > A$ by,

$$\mathcal{L}(f) = F(s) \ := \ \int_0^\infty e^{-st}\, f(t)\ dt\,.$$

Since for all t the function e^{-st} is an analytic function of s one has the Cauchy-Riemann equations

$$\frac{d\, e^{-st}}{ds} \ = \ \frac{\partial e^{-st}}{\partial x} \ = \ \frac{1}{i}\frac{\partial e^{-st}}{\partial y} \qquad \text{where} \quad s = x + iy\,.$$

Differentiating under the integral sign shows that $F(s)$ also satisfies the C-R equations so is an analytic function in $\mathrm{Re}\, s > A$ with

$$F'(s) \ = \ \int_0^\infty \frac{d}{ds}\Big(e^{-st}\, f(t)\Big)\ dt \ = \ \int_0^\infty e^{-st}\,(-t)f(t)\ dt \ = \ \mathcal{L}(-tf)\,.$$

Inductively one has

$$F^{(n)} = \mathcal{L}\Big((-t)^n\, f\Big)\,.$$

An explicit integration shows that

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}\,.$$

Thus e^{at} is the inverse Laplace transform of $1/(s-a)$. The differentiation formula shows that

$$\mathcal{L}((-t)e^{at}) = \frac{-1}{(s-a)^2}\,.$$

Once more and one obtains

$$\mathcal{L}((-t)^2e^{at}) = \frac{(-1)(-2)}{(s-a)^3}\,.$$

By induction

$$\mathcal{L}\big(t^n\, e^{at}\big) = \frac{(n)!}{(s-a)^{n+1}}\,.$$

Thanks to the partial fraction decomposition of any rational function $P(s)/Q(s)$ with $\deg P < \deg Q$ as a finite linear combination of the functions $(s-a)^{-n}$ *one can compute the inverse Laplace Transform of any rational function once one knows the exact roots of the denominator.*