Open Mapping Theorem

1 Definiton and statement.

Definition 1.1. *A map from an open set $\Omega \subset \mathbb{C}$ to $\mathbb{C}$ is an* **open mapping** *when the image by f of any open subset of Ω is open.*

Proposition 1.1. *A map is open if and only if for each $z \in \Omega$ the image of any open set containing z contains a neighborhood of $f(z)$.*

Exercise 1.1. Prove this proposition.

Examples 1.1. 1. The map $f(z) = az + b$ with $a, b \in \mathbb{C}$ is open when $a \neq 0$ and not open when $a = 0$.

2. The map

$$x + iy \mapsto \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} (x, y)$$

is open when the matrix M is invertible and not open otherwise. The same is true of $a\overline{z} + b$.

3. If f is analytic with $f'(z) \neq 0$ at all z then f is locally invertible by the inverse function theorem so satisfies the criterion of Proposition 1.1.

4. The mapping $f(z) = z^k$ with $k \geq 1$ integer is open. To prove this it suffices to show that the image by f of a small open set about $z = 0$ contains a neighborhood of 0. For that it suffices to observe that the image by f of $\{|z| < r\}$ is exactly the disk of radius $r^{1/k}$. Each point in the latter disk has k preimages in the former disk located at the vertices of regular k-gon.

5. The map $x + iy \mapsto x^2 + iy$ is not open. The image of the open set $\mathbb{C}$ is not open. The restriction of this map to any set Ω that does not meet the imaginary axis is an open mapping. The mapping is not analytic.

Proposition 1.2. *The composition of open maps is open. Precisely, if g is open on Ω_1, f is open on Ω_2 and $g(\Omega_1) \subset \Omega_2$ then $f\big(g(z)\big)$ is open on Ω_1.*

Exercise 1.2. Prove this proposition.

Theorem 1.3. *If $f(z)$ is a nonconstant analytic on an open connected set Ω, then f is an open mapping.*

2 Proof of the Open Mapping Theorem.

Proof. Verify the condition of Proposition 1.3. At $\underline{z} \in \Omega$ where $f'(\underline{z}) \neq 0$ the map is locally invertible and the condition is automatic.

Suppose on the other hand that $\underline{z} \in \Omega$ and $f'(\underline{z}) = 0$. Since f is not constant there is a smallest $k \geq 2$ so that $f^{(k)}(\underline{z}) \neq 0$. Taylor's theorem yields for z near $\underline{z}$

$$f(z) - f(\underline{z}) \;=\; c_k(z-\underline{z})^k + c_{k+1}(z-\underline{z})^{k+1} + \cdots \qquad c_k \neq 0\,.$$

Factor to find

$$f(z) = c_k(z-\underline{z})^k\Big(1 + a_1(z-\underline{z}) + a_2(z-\underline{z})^2 + \cdots\ \Big) \;:=\; c_k(z-\underline{z})^k h(z)\,,$$

with

$$h(z) \;:=\; 1 + a_1(z-\underline{z}) + a_2(z-\underline{z})^2 + \cdots\,.$$

Then h is analytic as the sum of a convergent power series.

Define $g(w)$ on a neighborhood of $w = 1$ to be a local inverse of the map $w = z^k$ on a neighborhood of $z = 1$ where the derivative $w'(1) = k \neq 0$. Then $g'(1) = 1/k$, $g(h(z))^k = h(z)$ for z near 1, and

$$f(z) - f(\underline{z}) \;=\; c_k\Big((z-\underline{z})g(h(z))\Big)^k\,. \tag{2.1}$$

Since

$$\frac{d}{dz}\Big((z-\underline{z})g(h(z))\Big)\Big|_{z=\underline{z}} = g(h(\underline{z})) = 1 \neq 0\,,$$

is follows that the map $z \mapsto (z-\underline{z})g(h(z))$ is open on a neighborhood of $\underline{z}$. Therefore equation (2.1) expresses f as the composition of the mappings $(z-\underline{z})g(h(z))$, z^k and $z \mapsto c_k z + f(\underline{z})$. Thus the image by f of an open set containing $\underline{z}$ contains an open neighborhood of $f(\underline{z})$ verifying the criterion of Proposition 1.3 at points where f' vanishes ☐

Remark 2.1. Examining (2.1) one sees that the preimage of a point $w \approx f(\underline{z})$ consists of k points near $\underline{z}$ nearly positioned at the vertices of a regular k-gon centered at $\underline{z}$. In this sense the behavior of $f - f(\underline{z})$ near $\underline{z}$ is well modelled by the k to one open mapping $c_k(z-\underline{z})^k$.