Math 558, Fall 2009 Prof. J. Rauch

Final Exam December 18, 2009

Instructions. 1. Four sides of a 3.5in. $\times$ 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ... etc.
3. Show work and explain clearly.
4. There are 7 questions. 58 points total.

1. (4+3) points). **a.** For the system

$$x' \ = \ y, \qquad y' = -2\,x$$

find a conserved quantity (a.k.a. integral of motion).
b. Use the conserved quantity to find an equation of and sketch the orbit through $(1,1)$.

Solution. a. Multiply the first equation by $2x$ and the second by y. Add the results to find

$$2x\,x' + y\,y' \ = \ 0.$$

This proves that $x^2 + y^2/2$ is constant on orbits.
Alternative strategy. One can show that the system is hamiltonian. One can find $H(x,y)$ so that

$$y = \frac{\partial H}{\partial y}, \qquad -2x = -\frac{\partial H}{\partial x}\,.$$

The general solution is $H = x^2 + y^2/2 +$ constant, recovering the integral of motion.

b. The orbits are

$$x^2 + \frac{y^2}{2} \ = \ C\,.$$

The orbit containing $(1,1)$ must have

$$1^2 + \frac{1^2}{2} \ = \ C\,.$$

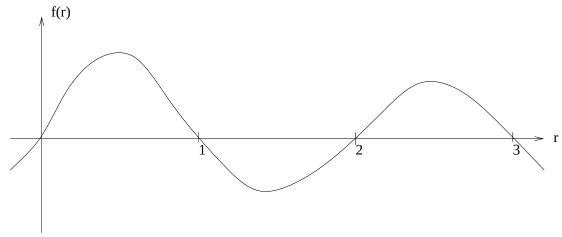
Therefore the orbit has equation

$$x^2 + \frac{y^2}{2} \ = \ \frac{3}{2}\,.$$

2. (5+3 points). Consider the system of ordinary differential equations in polar coordinates,

$$r' = f(r), \qquad \theta' = \phi(r) \geq \delta > 0, \qquad r > 0.$$

The function f is bounded, is negative outside $[0,3]$, and has graph indicated below.



a. Sketch the orbits indicating all periodic orbits.

b. For each periodic orbit determine if it is stable, asymptotically stable, or unstable. The notions of stability are in the orbital sense.

Solution. The circles of radius 1,2,3 are periodic orbits traversed in the clockwise sense since $\theta' > 0$.

The orbits in $0 < r < 1$ spiral out toward the unit circle since $f > 0$ there.

The orbits in $1 < r < 2$ spiral in toward the unit circle since $f < 0$ there.

The orbits in $2 < r < 3$ spiral out toward the circle of radius 3 since $f > 0$ there.

Orbits in $r > 3$ spiral in toward $r = 3$ since $f < 0$ there.

The orbits $r = 1$ and $r = 3$ are asymptotically stable while the orbit $r = 2$ is unstable.

3. (5+4 points). $\lambda = 1$ is an eignevalue of

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

a. Find the generalized eigenspace, X_j, associated to this eigenvalue.

b. For $\underline{x} \in X_j$ find the solution of

$$X' = A\,X, \qquad X(0) = \underline{x}\,.$$

Solution. Since $A - \lambda I$ is upper triangular its determinant is the product of the diagonal elements,

$$\det(A - \lambda I) \;=\; (\lambda - 1)^3(\lambda - 2)\,.$$

Therefore the generalized eigenspace has dimension $d = 3$.

Compute,

$$A - I \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (A - I)^2 \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The kernel of the latter is $\{(x_1, x_2, x_3, 0)\}$ has dimension three so is the generalized eigenspace. In a typical case with multiplicity three, the generalized eigenspace is the kernel of $(A - \lambda I)^3$ and is larger than the nullspace of $(A - \lambda I)^2$. In **the present example** the nullpsaces of $(A - I)^2$ and $(A - I)^3$ are the same and $(A - I)^2 = 0$ on X_j. The solution with initial datum $\underline{x} \in X_j$ is

$$e^{\lambda t}\, e^{(A - \lambda I)t}\, x \;=\; e^t \Big[I \;+\; (A - I)t \Big] \underline{x}\,.$$

Here is a common error. It is true that

$$e^{At} = e^{\lambda t} e^{(A - \lambda I)t} \quad \text{and,} \quad e^{(A - \lambda_j I)t}\underline{x} = \Big[I + (A - \lambda_j I)t + \cdots + \frac{\big((A - \lambda_j I)t\big)^{m_j - 1}}{(m_j - 1)!} \Big] \underline{x} \quad \text{for } \underline{x} \in X_j.$$

It is **NOT TRUE** that

$$e^{(A - \lambda I)t} = I + (A - \lambda_j I)t + \cdots + \frac{\big((A - \lambda_j I)t\big)^{m_j - 1}}{(m_j - 1)!}\,.$$

4. (5 points). The periodic linear equation

$$X' = A(t)\,X\,, \qquad A(t+\omega) = A(t)\,,$$

has fundamental matrix Ψ with

$$\Psi(0) = I, \qquad \Psi(\omega) = \begin{pmatrix} 2 & 10 \\ 0 & -0.5 \end{pmatrix}.$$

Is the solution $X = 0$ stable? Explain.

Solution. A neccessary condition for stability is that the eigenvalues of $\Psi(\omega)$ have modulus ≤ 1. Since Ψ has eigenvalue $\lambda = 2$ which is larger than one, the solution $X = 0$ is unstable.

5. (6 points) (Hirsh-Smale-Devaney 212/6) Find a strict Lyapunov function for the equilibrium point $(0,0)$ of the the system,

$$x' = -2x - y^2, \qquad y' = -y - x^2 .$$

Find a $\delta > 0$ (larger is better) so that the disk of radius δ and center $(0,0)$ is contained in the basin of attraction of $(0,0)$.

Solution. Try $L := x^2 + y^2$ as Lyapunov function. Then

$$\dot{L} = x\dot{x} + y\dot{y} = x(-2x - y^2) + y(-y - x^2) = -2x^2 - y^2 - xy^2 - yx^2 .$$

So

$$\dot{L} \le -L - xy^2 - yx^2 . \qquad \text{(3 points)}$$

On, $x^2 + y^2 \le \delta^2$ one has

$$|xy^2| \le \delta L, \qquad |yx^2| \le \delta L .$$

So,

$$\dot{L} \le -L + 2\delta L = -(1 - 2\delta)L .$$

If $\delta < 1/2$ then $\dot{L} < 0$. More precisely for if $L((x(0), y(0)) = \delta^2 < 1/4$ $L((x(t), y(t)) \le e^(1 - 2\delta)t\, L((x(0), y(0))$ so the disk of radius $\delta < 1/2$ is in the basin of attraction.

Better solution. For the same L

$$\dot{L} = -(1 + x)y^2 - (2 + y)x^2 .$$

Let Q denote the open quadrant containing $(0,0)$ in its interior,

$$Q := \{(x, y) \ : \ x > -1, \text{ and, } y > -2\} .$$

Then $\dot{L} < 0$ on $Q \setminus (0,0)$. Since the unit disk $\{L < 1\}$ is contained in Q it follows that this disk is in the basin of attraction by LaSalle's Invariance Prinicpal applied to each of the compact invariant sets $\{L \le \delta^2 < 1\}$.

6. (6+4+5 points). Consider the gradient system

$$X' \;=\; -\nabla_X V(X)\,, \qquad V(x_1\,,\,x_2) = \sin x_1\, \sin x_2\,.$$

$V(x_1\,,\,x_2) = \sin x_1 \sin x_2$ has critical points ($\nabla V = 0$) at $(n\pi, m\pi)$ and $(\pi/2\,,\,\pi/2) + (n\pi, m\pi)$ where n, m are integers. The critical points are maxima ($V = 1$), mountain passes ($V = 0$), and minima ($V = -1$).

a. Show that the equilibrium $(0,0)$ is a saddle by linearizing. (An equilibrium is called a saddle when the linearization has a saddle.) Find the tangent directions of the unstable and stable manifolds. *Ans.* $(1,-1)$ and $(1,1)$.

b. Show that the minimum of V at $(\pi/2, -\pi/2)$ is a strict local minimum.

Do not prove the next fact. In this way one can show that all the maxima and minima are strict local extrema.

c. Use Lyapunov's method to show that the equilibria which are minima are asymptotically stable and that the maxima are unstable.

Solution. a. The linearization is

$$Z' \;=\; -\begin{pmatrix} \frac{\partial^2 V(0,0)}{\partial^2 x_1} & \frac{\partial^2 V(0,0)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V(0,0)}{\partial x_2 \partial x_1} & \frac{\partial^2 V(0,0)}{\partial^2 x_2} \end{pmatrix} Z \;=\; \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} Z \;:=\; AZ\,. \qquad \text{2 points}$$

The eigenvalues of A are $\lambda = \pm 1$. Since there is one positive and one negative, the linearization is a saddle so the equilibrium is a saddle. 1 point

The eigenspace corresponding to eigenvalue $\lambda = 1$ is the nullspace of

$$A - I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}. \qquad \text{1 point}$$

The eigenvectors are multiples of $(1, -1)$. Since A is symmetric the other eigenvectors are perpendicular hence multiples of $(1, 1)$. The tangent direction of the unstable manifold is $(1, -1)$ and the tangent direction of the stable manifold is $(1, 1)$. 1+1 point

b. The matrix of second derivatives at the minimum $(\pi/2, -\pi/2)$ is equal to

$$\begin{pmatrix} \frac{\partial^2 V(\pi/2,-\pi/2)}{\partial x_1^2} & \frac{\partial^2 V(\pi/2,-\pi/2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V(\pi/2,-\pi/2))}{\partial x_2 \partial x_1} & \frac{\partial^2 V(\pi/2,-\pi/2))}{\partial x_2^2} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \text{2 points}$$

Since this symmetric matrix has strictly positive eigenvalues we know that V has a strict local minimum. 2 points

Alternate solution. V is a product of sines so has values in $[-1, 1]$. The $V(0, 0) = -1$ so is a global minimum. To show it is strict it suffices to note that if $0 < |x_1 - \pi/2| << 1$ then $\sin x_1$ is close to but less than 1. Similarly for $0 < |x_2 - (-\pi/2)| << 1$, $\sin x_2$ is close to but larger than -1. It follows that if (x_1, x_2) is close to but not equal to $(\pi/2\,,\,\pi/2)$ then V is close to but larger than -1.

c. The function V is a strict Lyapunov function for the minima and it follows that they are asymptotically stable. 3 points

In the same way $\dot V < 0$ near the maxima and one concludes that orbits on such a neighborhood approach the maximum *as t tends to $-\infty$*. Therefore the equilibrium is unstable. 2 points

7. (4+4 points). This problem continues the study of the gradient system in problem 6.

a. Show that on the line $x_1 = x_2$, the direction of motion is parallel to $(1, 1)$.

Do not prove the next facts. It follows that the slope one line through the origin is invariant under the flow. The same reasoning applies to all lines with slope ± 1 and passing through the saddle points.

b. The two orbits approaching the saddle at $(0, 0)$ along the stable curve (tangent direction $(1, 1)$ from **6b**) approach what points as $t \to -\infty$? What is the value of V at those points?

Solution. a. The direction is

$$-\left(\frac{\partial V(x_1, x_2)}{\partial x_1}, \frac{\partial V(x_1, x_2)}{\partial x_2}\right) \;=\; -(\cos x_1 \sin x_2 \,,\, \sin x_1 \cos x_2)\,.$$

When $x_1 = x_2$ this is equal to

$$-(\cos x_1 \sin x_1 \,,\, \sin x_1 \cos x_1) \;=\; -\cos x_1 \,\sin x_1\,(1, 1)\,,$$

which is parallel to $(1, 1)$.

b. The values of V decrease on orbits. So the orbits arriving along stable directions have values that decrease. From **7a** the orbit stays on the line of slope 1. They come from the first equilibrium on the line. Those are the points $\pm(\pi/2, \pi/2)$. They are points where V attains its maximum values.

In the same way, the unstable curves approach neighboring minima on the line of slope -1. This was not asked in the question.