Math 558, Fall 2009 Prof. J. Rauch

Midterm Exam October 22, 2009

Instructions. 1. Two sides of a 3.5in. × 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, . . . etc.
3. Show work and explain clearly.
4. There are five questions, one per page. 70 points total.

1. (10 points). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, $\infty < a < b < \infty$, and $f(a) = f(b) = 0$. Show that if $x(t)$ is a solution of

$$x' \ = \ f(x),$$

with

$$a < x(0) < b,$$

then for all $t \geq 0$,

$$a \ < \ x(t) \ < \ b\,.$$

Bargain. Full credit granted it you show that $x(t) < b$.

Solution. If the inequality $x(t) < b$ is violated there would be a $T > 0$ with $x(T) \geq b$. By continuity of $x(t)$ there would therefore be a $0 < \underline{t} \leq T$ with $x(\underline{t}) = b$. Then $x(t)$ is a solution of the initial value problem

$$x' \ = \ f(x)\,, \qquad x(\underline{t}) = b\,.$$

The function $y(t) = b$ is also a solution of this initial value problem. *The uniqueness theorem for solutions of the intial value problem* implies that $x = y$ so $x(t) = b$ for all t. Since $x(0) \neq b$ this is impossible and therefore it cannot be that $x(T) \geq b$.

Prob.	Score
1	
2	
3	
4	
5	
Sum, %	

2. (15 points). For the initial value problem,

$$x' - x + \epsilon\, x^2 = 0, \qquad x(0) = 1, \qquad 0 \le \epsilon << 1,$$

Compute the first two terms of first order Taylor approximation (a.k.a. perturbation theory of first order),

$$x \;\approx\; x_0(t) + \epsilon x_1(t)\,.$$

Solution. Denote the solution by $x(t, \epsilon)$. *The fundamental existence theorem implies that $x(t, \epsilon)$ is an infinitely differentiable differentiable function of t, ϵ.* Need $x_0(t) = x(t, 0)$ and $x_1(t) = \partial_\epsilon x(t, 0)$.

Setting $\epsilon = 0$ in the initial value problem yields,

$$x_0' - x_0 \;=\; 0\,, \qquad x_0(0) \;=\; 1\,. \qquad \text{3 points}$$

Solve to find,

$$x_0(t) \;=\; e^t\,. \qquad \text{2 points}$$

Differentiating with respect to ϵ, the equations in the initial value problem yields,

$$(\partial_\epsilon x)' - \partial_\epsilon x + x^2 + 2\,\epsilon\, x\, \partial_\epsilon x \;=\; 0\,, \qquad \partial_\epsilon x(0, \epsilon) \;=\; 0\,. \qquad \text{5 points}$$

Setting $\epsilon = 0$ and using the formula for x_0 yields

$$x_1' - x_1 + e^{2t} \;=\; 0, \qquad x_1(0) \;=\; 0\,. \qquad \text{2 points}$$

Solving yields,

$$x_1 = -e^{2t} + e^t\,. \qquad \text{3 points}$$

So,

$$x \;\approx\; e^t \;+\; \epsilon(-e^{2t} + e^t)\,.$$

3. (15 points). For

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

a. Compute the characteristic polynomial $\det(A - zI)$.

b. Find the eigenvalues λ_j.

c. Find all the eigenvectors.

d. Find the associated generalized eigspaces X_j.

e. Find e^{At}.

Solution.

$$\det(A - zI) = (z-2)^4 . \qquad \text{3 points}$$

The only eigenvalue is $\lambda = 2$. 1 point

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

This matrix has kernel of dimension equal to 2 given by the equations $x_2 = 0$ and $x_3 = 0$. The kernel consists of vectors $(x_1, 0, 0, x_4)$. These are the eigenvectors. 4 points

$$(A - 2I)^2 = 0$$

has kernel of dimension equal to 4. So the generalized eigenspace is $\mathbb{C}^4$. 3 points

$$e^{At} = e^{2t}\big(I + t(A - 2I)\big) = e^{2t} \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t & 1 \end{pmatrix} . \qquad \text{4 points}$$

Remark. If you make an arithmetic error and compute an incorrect eigenvalue, then when you seek an eigenvector you will find that $X = 0$ is the only solution. This alerts you to your error and you can go back and fix it. If there is no eigenvector *do not* just go forward.

4. (15 points). **a.** For the system

$$X' \;=\; \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} X \;=\; 0\,,$$

determine the type of the phase portrait, *i.e.* saddle, spiral sink, ... *etc.*

b. If a saddle determine the stable and unstable lines. If a center or spiral, determine the axes of the underlying ellipse and the direction of rotation. If a node determine the limiting slope as $t \to +\infty$ and $t \to -\infty$.

Solution.

$$\det(A - zI) \;=\; \det\begin{pmatrix} 3-z & 5 \\ -2 & -3-z \end{pmatrix} \;=\; z^2 - 9 + 5\cdot 2 \;=\; z^2 + 1\,.$$

The eigenvalues are $\pm i$. Distinct complex conjugate purely imaginary roots makes this a **center**.

2+2 points

The direction of rotation is determined by examining the tangent direction $A(1,0) = (3,-2)$ when $X = (1,0)$. The second component -2 is negative, so the direction is downward showing that the rotation is **clockwise**.

4 points

The direction of the axes of the ellipses are determined by

$$0 \;=\; AX\cdot X \;=\; \Big(3x_1 + 5x_2\,,\, -2x_1 - 3x_2\Big)\cdot\Big(x_1, x_2\Big) \;=\; 3x_1^2 \;+\; (5-2)x_1x_2 \;-\; 3x_2^2 \quad \text{4 points}$$

This equation has no solution with $x_2 = 0$. Dividing by x_2^2 shows that the solutions are equal to $x_2^2\,\big(\eta, 1\big)$ where,

$$\eta = x_1/x_2 \qquad \text{satisfies,} \qquad \eta^2 + \eta - 1 = 0.$$

The roots are

$$\eta \;=\; \frac{-1 \pm \sqrt{1 - 4(-4)}}{2} \;=\; \frac{-1 \pm \sqrt{5}}{2}\,.$$

The direction of the axes are the orthogonal directions

$$\Big(\frac{-1+\sqrt{5}}{2}\,,\,1\Big), \qquad \Big(\frac{-1-\sqrt{5}}{2}\,,\,1\Big)\,. \qquad \text{3 points}$$

Remark. These vectors are of unequal lengths both longer than one. They could be normalized to length one, but the question does not ask for that. Their relative lengths do NOT correspond to the relative lengths of the axes of the associated ellipses. For those lengths see my recently revised web posting on the ellipses.

5. (15 points). For the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{pmatrix},$$

a. Compute the characteristic polynomial $\det(B - zI)$.

b. Find the eigenvalues λ_j.

c. Find the eigenvectors.

d. Find the associated generalized eigspaces X_j.

e. Find the general solution of

$$X' = BX.$$

Solution.

$$\det(B - zI) = (z-2)(z-3)^2.$$

The eigenvalues are 2 and 3. 3 points

The eigenvalue 2 has multiplicity one. The generalized eigenspace is

$$\ker(B - 2I) = \mathbb{C}(1,0,0).$$

The nonzero elements are the eigenvectors. A one dimensional set of solutions corresponding to this root of multiplicity one is

$$a\, e^{2t}\,(1,0,0).$$ 3 points

$$B - 3I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

The eigenvectors are the nonzero elements of,

$$\ker(B - 3I) = \mathbb{C}\,(0,0,1).$$ 3 points

$$(B - 3I)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \ker(B - 3I)^2 = (0,b,c).$$

That two dimensional space is the generalized eigenspace. 3 points

The exponential formula shows that the solutions with initial data in this space have the form

$$e^{3t}(I + (B - 3I)t)(0,b,c).$$

The general solution is,

$$a\, e^{2t}\,(1,0,0) + e^{3t}(I + (B - 3I)t)(0,b,c).$$ 3 points