Math 558, Fall 2010 Prof. J. Rauch

Final Exam Solutions December 14, 2010

Instructions. 1. Two sides of a 3.5in. $\times$ 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, …etc.
3. Show work and explain clearly.
4. There are 7 questions, seven pages, and a total of 74 points.
5. You may use the back of the pages and/or supplementary sheets.

1. (2+5+3+3 points). **i.** Explain why $e^{A+cI} = e^c e^A$ for all scalars c and matrices A.
ii. Compute e^{At} for all $t \in \mathbb{R}$, where

$$A := \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad \omega > 0\,.$$

iii. Write the formula for the solution of the initial value problem

$$X' = \begin{pmatrix} -1 & \omega \\ -\omega & -1 \end{pmatrix} X + f(t), \qquad X(0) = 0$$

using the variation of parameters formula and the results above.
iv. Show that if $f(t)$ is a continuous bounded function * on $0 \le t < \infty$, then the solution from **iii** is also a bounded function on $0 \le t < \infty$. **Discussion.** This is the sort of thing variation of parameters is good for.

Solution. i. One has $e^{A+B} = e^A\, e^B$ whenever $AB = BA$. Appliy with $B = cI$ together with $e^{cI} = e^c\, I$. The latter identity follows immediately from the series definition,
ii. $\det(A - \lambda I) = \lambda^2 + \omega^2$ so the eigenvalues are $\pm i\,\omega$.
The eigenvectors for the plus sign are the nonzero elements of the kernel of the singular matrix,

$$A - (i\,\omega)I = \begin{pmatrix} -i\omega & \omega \\ -\omega & -i\omega \end{pmatrix} = \omega \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}.$$

The kernel is defined by the equation $x_2 = ix_1$ so is spanned by $(1, i)$. Therefore $e^{i\omega t}(1, i)$ is a solution. It together with its complex conjugate generate the general solution,

$$a_+\, e^{i\omega t}(1, i) + a_-\, e^{-i\omega t}(1, -i)\,, \qquad a_+, a_- \in \mathbb{C}\,.$$

Probl.	1	2	3	4	5	6	7	Sum	%
Points	13	10	11	10	9	12	9	74	100

* **Definition.** *A function* $f : [0, \infty[\to \mathbb{C}^N$ *is bounded when there is an* $M > 0$ *so that for all* $t \ge 0$, $\|f(t)\| \le M\,.$

The solution with $X(0) = (1,0)$ is the first column of e^{At}. The initial value $(1,0)$ is attained by the choice $a_+ = a_- = 1/2$.

$$X(t) \;=\; \frac{1}{2}\Big(e^{i\omega t}(1,i) + e^{-i\omega t}(1,-i)\Big) \;=\; \big(\cos\omega t\,,\,-\sin\omega t\big)\,.$$

The solution with $X(0) = (0,1)$ is the second column of e^{At}. The initial value $(0,1)$ is attained by the choice $a_+ = -a_- = 1/2i$,

$$X(t) \;=\; \frac{1}{2i}\Big(e^{i\omega t}(1,i) - e^{-i\omega t}(1,-i)\Big) \;=\; \big(\sin\omega t\,,\,\cos\omega t\big)\,.$$

So,

$$e^{At} \;=\; \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Alternate. One can compute explicitly the powers of A^n and sum the series defining e^{At} recognizing the series for sine and cosine in the answer. The advantage of the method presented above is that it works in general. It is rare for the series for sine and cosine to pop out so nicely.

iii. Use the variation of constants formula

$$X(t) \;=\; \int_0^t e^{(A-I)(t-s)}\; f(s)\; ds \;=\; \int_0^t e^{-(t-s)}\; e^{A(t-s)}\; f(s)\; ds\,.$$

iv. There is a constant C independent of t, s so that

$$\big\|e^{A(t-s)}\; f(s)\big\| \;\le\; C\,\|f(s)\|\,.$$

Therefore

$$\|X(t)\| \;\le\; \int_0^t e^{-(t-s)}\; C\,\|f(s)\|ds \;\le\; C\,\sup_{s\ge 0}\|f(s\|\int_0^t e^{-(t-s)}\; ds\,.$$

The final integral is no larger than $\int_0^\infty e^{-\sigma}\,d\sigma = 1$ so the right hand side gives a uniform bound on the values of X.

2. (2+5+1+2 points). (Hale/Kocak 276/1). **i.** For any real a find the unique equilibrium point of Lienard's equation

$$\dot{x}_1 \;=\; x_2 - a(x_1^3 - x_1)\,, \qquad \dot{x}_2 \;=\; -x_1\,.$$

ii. Determine the stability of the equilibrium for those values of a where stability is determined from the linearization.

iii. The phase portrait *of the linearization at the equilibrium* changes type when a changes from negative to positive. Describe the change in the phase portrait.

iv. In addition to the conclusion of *iii*, what change in the phase portrait of the *nonlinear* system do you expect when a changes from negative to positive?

Solution. i. Solve the pair of equations,

$$x_2 - a(x_1^3 - x_1) \;=\; 0, \qquad -x_1 \;=\; 0\,.$$

Plugging the second into the first yields $x_2 = 0$ so the unique equilibrium is $(0,0)$.

ii. The linearization at the origin arises by dropping the higher order terms to yield the linear system

$$Y' \;=\; \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} \;:=\; A\,X\,.$$

Compute

$$\det(A - \lambda I) \;=\; \det\begin{pmatrix} a-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \;=\; -\lambda(a-\lambda) + 1 \;=\; \lambda^2 - a\lambda + 1\,.$$

Setting this equal to zero and using the quadratic formula yields the eigenvalues,

$$\lambda \;=\; \frac{a \;\pm\sqrt{a^2-4}}{2}\,.$$

For $|a| \geq 2$ there are two real roots with the same sign as a. For $|a| < 2$ there are a pair of complex conjugate roots whose real part is equal to $a/2$. The linearization is a sink for $a < 0$ and a source for $a > 0$. *The nonlinear system is asymptotically stable for $a < 0$ and unstable for $a > 0$.*

iii. When a crosses zero from negative to positive, a pair of complex conjugate roots cross from the left half plane to the right. The phase portrait changes from a spiral sink to a spiral source.

iv. This is an example of a *Hopf bifurcation* where the equilibrium at the origin becomes unstable and a periodic orbit is generated at the origin.

3. (1+3+3+4 points). Consider the damped hard spring equation

$$x'' \; + \; a\, x' \; + \; x \; + \; x^3 \; = \; 0$$

where the coefficient of friction is $a \geq 0$.

i. Write the equation as a first order system.

ii. Find a potential energy function $V(x)$ so that when $a = 0$ the energy $\dot{x}^2/2 + V(x)$ is a conserved quantity. Derive an energy dissipation identity for $a \geq 0$.

iii. Use the energy function to show that the equilibrium $x = x' = 0$ is stable for all $a \geq 0$. State clearly the theorem you are using. **Discussion.** The interest is the case $a = 0$ which cannot be obtained by linearization.

iv. Show that for $a > 0$, **every** solution converges as $t \to \infty$ to the equilibrium at the origin. **Hint.** Lasalle Invariance together with the energy dissipation law. State clearly the hypotheses that you verify.

Solution. i. Introduce $v := \dot{x}$. The equation is equivalent to the system

$$\dot{x} \; = \; v\,, \qquad \dot{v} \; = \; -a\, v - x - x^3\,.$$

ii. Want $-dV/dx = -x - x^3$ so $V = x^2/2 + x^4/4 + C$. In the sequel we take the constant $C = 0$. Then

$$\frac{d}{dt}\Big(\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{x^4}{4} \Big) \; = \; x' x'' + x\, x' + x^3 \dot{x} \; = \; x' \Big(x'' \; + \; x \; + \; x^3 \Big)\,.$$

The first conclusion is that when $a = 0$ the derivative vanishes and we have a conserved quantity. The second conclusion is that solutions for general $a \geq 0$ satisfy,

$$\frac{d}{dt}\Big(\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{x^4}{4} \Big) \; = \; -a (x')^2\,. \tag{1}$$

iii. Introduce the Lyapunov function

$$L(x, v) \; := \; \frac{v^2}{2} + \frac{x^2}{2} + \frac{x^4}{4}\,.$$

Then $L > 0$ except at the origin so the origin is a strict minimum. The preceding computation shows that for $a \geq 0$, $\dot{L} = -a v^2 \leq 0$. Stability of the origin follows by Lyapunov's Theorem.

iv. For any constant $R > 0$ define

$$\mathcal{P} \; := \; \Big\{ (x, v) \; : \; L(x, v) \leq R \Big\}\,.$$

Then $\mathcal{P}$ is closed because L is continuous and it is bounded since $x^2 + v^2 \leq 2R$ for points of $\mathcal{P}$. Since L decreases on orbits $\mathcal{P}$ is postively invariant. To apply Lasalle's invariance principle we need to show that the $x = v = 0$ is the only orbit in $\mathcal{P}$ along which L is constant. Suppose that $x(t), v(t)$ is an orbit in $\mathcal{P}$ along which L is constant. From (1) together with $a > 0$ it follows that $x' = 0$ so $x(t)$ is constant. The differential equation then implies that

$$0 = x + x^3 = x(1 + x^2)\,.$$

Since $1 + x^2 > 0$ it follows that $x = 0$ so the orbit is $x = v = 0$. LaSalle implies that every orbit in $\mathcal{P}$ tends to $(0, 0)$ as $t \to \infty$.

For any point $\underline{x}, \underline{v}$ one can choose $R = L(\underline{x}, \underline{v})$ then $\underline{x}, \underline{v} \in \mathcal{P}$. Therefore the orbit through $\underline{x}, \underline{v}$ tends to $(0, 0)$.

4. (5+5 points). **i.** In polar coordinates consider the system

$$r' \;=\; f(r,\theta)\,, \qquad \theta' \;=\; g(r,\theta) > 0 \quad \text{in} \quad r > 0\,.$$

Suppose that f is continuously differentiable and

$$f > 0 \quad \text{when} \quad r = 1\,, \qquad \text{and} \qquad f < 0 \quad \text{when} \quad r = 2\,.$$

Prove that every orbit starting on the circle $\{r = 1\}$ spirals to its ω-limit set which is a periodic orbit. **Hint.** Use a result from the course to make this easy.

ii. a. Sketch an example satisfying these hypotheses where the periodic limit is asymptotically stable.

b. Sketch an example satisfying these hypotheses where the periodic limit is unstable.

c. Sketch an example satisfying these hypotheses where the periodic limit is stable but not asymptotically stable.

Solution. i. Consider the annulus

$$\mathcal{A} \;:=\; \big\{1 \;\leq\; r \;\leq\; 2\big\}\,.$$

Since $f > 0$ on $r = 1$ it follows that the direction field is transverse to and points into $\mathcal{A}$ along the inner circular boundary $r = 1$.

Since $f < 0$ on $r = 2$, the direction field is transverse to the boundary and inward pointing on outer circular boundary $r = 2$. It follows that $\mathcal{A}$ is a positively invariant set.

Since $\theta' > 0$ in $\mathcal{A}$ it follows that there is no equilibrium in $\mathcal{A}$.

The Poincaré-Bendixon Theorem implies that the the orbit through any point in $\mathcal{A}$ sprials to a periodic orbit.

ii. a. Take an example where the orbit spirals to $r = 3/2$ and in addition, obits with $r > 3/2$ spiral in to $r = 3/2$. If you consider the section $\{\theta = 0\}$, the Poincaré map has the unique fixed point at $r = 3/2$ and orbits inside move monotonically out while those outside move monotonically in.

b. Again take a unique fixed point of the Poincaré map with orbits moving monotonically away for $r > 3/2$. In this case orbits in $r > 3/2$ spiral away from the periodic orbit.

c. Take a Poincaré map for which $[4/3\,,\,5/3]$ consists of fixed points and the map is strictly monotone increasing in $[1, 4/3[$ and strictly monotone decreasing in $]5/3\,,\,2]$. There is an inner annulus of periodic orbits. Orbits on either side of the core spiral to the core. Periodic orbits in the inner annulus are stable and not asymptotically stable.

5. (3+3+3 points). Consider the scalar equation

$$x' \;=\; x\, g(x,a)$$

depending on a real parameter a. The equilibria consist of the line $\{x=0\}$ and the level curve $\{g=0\}$. Suppose that g is infinitely differentiable with

$$g(0,0)=0\,, \qquad \text{and,} \qquad \big(\, g_x(0,0)\,,\, g_a(0,0)\,\big) \;\neq\; (0,0)\,.$$

i. Give sufficient conditions involving partial derivatives of g that guarantee that near $(0,0)$ the level set $\{g=0\}$ is a graph of a smooth function $a=h(x)$.

ii. Give additional sufficient conditions involving partial derivatives of g that guarantee that near $(0,0)$ the level set $\{g=0\}$ is tangent to the x-axis, $\{a=0\}$, at the origin.

iii. Give additional sufficient conditions involving partial derivatives of g that guarantee that near $(0,0)$ the level set is strictly convex and lies in $a\geq 0$. **Discussion.** This is called *supercrtical bifurcation.*

Solution. i. The Implicit Function Theorem asserts that near $(0,0)$ the level set is a smooth curve since $\nabla_{a,x}g(0,0)\neq 0$. The normal to the curve is parallel to $\nabla_{a,x}g(0,0)$. The level set is a graph $a=h(x)$ when that normal is NOT parallel to the x axis. It is parallel when $g_a=0$. *The level set is a graph $a=h(x)$ when $g_a(0,0)\neq 0$.*

ii. In this case compute the derivative by differentiating $g(x,h(x))=0$ to find

$$g_x \;+\; g_a\, h' =\; 0, \qquad h' \;=\; -g_x/g_a\,. \tag{2}$$

When $\nabla_{a,x}g(0,0)\neq 0$, the curve is tangent to $\{x=0\}$ precisely when $h'=0$. *The curve is tangent to $\{a=0\}$ at $(0,0)$ if and only if $g_x(0,0)=0$.*

Alternate solution. One has tangency when and only when the normal is parallel to the x-axis. This holds when and only when $g_x(0,0)=0$.

iii. Compute the second derivative of h by differentiating the first identity in (2) with respect to a to find,

$$(g_a h'' + g_{aa}\,(h')^2 + g_{xa}\, h') + (g_{xa}\, h' + g_{xx}) \;=\; 0\,,$$

where the parentheses enclose the derivative of one or the other of the summands. Injecting the fact that $h'(0)=0$ yields

$$h''(0) \;=\; -g_{xx}(0,0)/g_a(0,0)\,. \tag{3}$$

The convexity of the question holds when $h''(0)>0$ which holds when $-g_{xx}(0,0)/g_a(0,0)>0$.

6. (3+3+3+(1+1+1) points). Consider the dynamics on $\mathbb{R}$ defined by iterating the map $x \mapsto x(x-a)$ with $0 < a < 1/2$.

i. Find all fixed points.

ii. Determine their stability.

iii. Determine the large n behavior of orbits $\{x_n : n \geq 0\}$ starting at arbitrary points x_0. Group them into sets that have the same behavior.

iv. Explain why this map has none of the three properties defining chaos, sensitive dependence, transitivity, and dense cycles.

Solution. i. x is a fixed point when and only when

$$x(x-a) = x, \qquad \text{equivalently} \qquad x(x-a-1) = 0\,.$$

There are two fixed points $x = 0$ and $x = a+1$.

ii. Set $f(x) := x(x-a) = x^2 - ax$. Compute

$$f' \;=\; 2x - a \qquad f'(0) = -a, \qquad f'(a+1) = 2(a+1) - a = a + 2\,.$$

Since $f'(a+1) > 1$, the equilibrium $a+1$ is unstable.

Since $|f'(0)| = |a| < 1$, the equilibrium $x = 0$ is stable.

iii. Remark. This part is much longer than I intended. It is a good study problem nevertheless.

Start at the right. In $]a+1, \infty[$, f is monotone increasing and $f(x) > x$ so orbits starting there increase monotonically to $+\infty$.

The point $a+1$ is fixed.

Next consider small x. Since $|f'(0)| < 1$ it follows that for x small, $|f(x)| < |x|$. For $x > 0$ this holds for $0 < x < a+1$ and fails at the right hand endpoint where $f(x) = x$. For negative x one has $|f(x)| < |x|$ until the root $x_l < 0$ of $f(x_l) = -x_l$. Find x_l by solving

$$x_l(x_l - a) = -x_l, \qquad x_l(x_l - a + 1) = 0, \qquad x_l = -1 + a\,.$$

On the interval $]-1+a\,,\,1+a[$ one has $|f(x)| < |x|$. The inequality becomes equality at the endpoints.

Since $|-1+a| < 1+a$ it follows that $\{|x| < 1-a\} \subset]-1+a, 1+a[$. *Orbits starting in $|x| < 1-a$ have modulus tending monotonically to zero.* Proof. We've shown the interval is invariant. An orbit $|x_n|$ satisfies $|x_n| \leq |x_0| < 1-a$. On $|x| \leq |x_0|$, $|f(x)/x|$ is continuous and < 1 so reaches a maximum $m < 1$. Therefore on the orbit

$$|x_{n+1}| = |f(x_n)| \leq m|x_n|\,.$$

By induction $|x_n| \leq m^n |x_0| \to 0$. ∎

Orbits starting in $[0, a]$ are mapped to $[-a^2/4, 0]$ since $-a^2/4$ is the minimum value of f. Since $a < 1/2$, $-a^2/4 > -1 + a$ so the succeeding points on the orbit have modulus tending to zero.

Next study orbits starting in $]a, a+1[$ where $0 < f(x) < x$. Orbits starting in this set decrease in a finite number of steps (the number depending on the initial condition) to a point in $]0, a]$ and then decrease in modulus to 0 by the preceding result.

We now know the future of all points starting in $x \geq 0$.

Points in $x < 0$ are immediately mapped to $x > 0$ and their future is determined. Define $X_l < 0$ to be the value so that $f(X_l) = 1 + a$. The solution in $x < 0$ is $X_l = -1$.

Points starting in $]-\infty, X_l[$ are mapped to $]a+1, \infty[$ and then increase to infinity.

X_l is mapped to $1 + a$ which is fixed.

Points in $]X_l, 0[$ have orbits tending to 0.

Summary. *The origin attracts orbits starting in* $]-1, a+1[$*. Positive* ∞ *attracts orbits starting in* $]-\infty\,, -1[\cup]a+1, \infty[$*.* 0 *and* $a+1$ *are fixed and* -1 *maps to* $a+1$*.*

iv. The equilibrium 0 is stable so points starting close stay close. In fact for $0 < \delta << 1$ the set $[-\delta, \delta]$ is postively invariant, so one *does not have sensitive dependence.*

The orbits starting in $x > a + 1$ increase to infinity. So the origin attracts no such orbit. The origin attracts all points in the invariant interval $[-\delta, \delta]$ for δ small. Therefore for all $n \geq 0$, $f^{-n}(]a+1, \infty[)$ does not meet $[-\delta, \delta]$. The dynamics *is not have transitive.*

For $\delta > 0$ and sufficiently small, orbits in $[-\delta, \delta]$ tend to zero. Therefore no cycle can meet $[-\delta, \delta]$. Therefore *the cycles are not dense.*

7. (3+1+3+2 points). **i.** Which of the following two systems is a gradient system, that is of the form $X' = -\text{grad}\, V(X)$?

$$\begin{aligned} &\textbf{a}. \quad x' \;=\; x^2 \;-\; 2xy\,, \qquad y' \;=\; y^2 \;-\; 2xy\,, \\ &\textbf{b}. \quad x' \;=\; x^2 \;-\; 2xy\,, \qquad y' \;=\; y^2 \;-\; x^2\,. \end{aligned}$$

ii. What is the relation of the integral curves of the gradient system and the level sets of V?

The next questions concern the gradient system with $V(x, y) := -(x - y^2)^2 + y^2$. The origin is an equilibrium that is a saddle point.

iii. Show that $V_y = 0$ on the x-axis. Show that the x-axis is invariant. Describe the flow on the x-axis.

iv. Use **iii** to explain why the unstable manifold of the origin is equal to the x-axis.

Solution. i. Write the systems as $x' = f$, $y' = g$. The system is gradient like when and only when $f_y = g_x$.

a. $f_y = -2x$, $g_x = -2y$. This is not gradient system.

b. $f_y = -2x$, $g_x = -2x$. This system is a gradient system.

ii. The integral curves of the gradient system are orthogonal to the level sets of V.

iii. $V_y = -2(x - y^2)(2y) + 2y$ so vanishes when $y = 0$. Since $V_y = 0$ on the x-axis the direction field is parallel to the axis. Therefore the x-axis is invariant.

Compute

$$V_x \;=\; -2(x - y^2)\,, \qquad V_x(x, 0) \;=\; -2x\,.$$

So the flow on the x-axis is given by $x' = 2x$. Precisely $(x(t), 0)$ is an integral curve if and only if $x' = 2x$.

iv. Each half of the x-axis is an orbit that converges to the origin as $t \to -\infty$. Thus the x axis is a curve of points whose orbits converge to the origin in the distant past. Since $(0, 0)$ is a saddle, the stable manifold theorem asserts that the unstable manifold is the union of two such orbits so must be equal to the x-axis.