Math 558, Fall 2010 Prof. J. Rauch

Midterm Exam October 21, 2010

Instructions. 1. Two sides of a 3.5in. $\times$ 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are five questions. Each is worth 10 points. You may use the back of the pages.

For the first two problems.

i. Find the general real or complex (your choice) solution.

ii. If the eigenvalues are real find all invariant real lines. Sketch them indicating their stability and other typical orbits.

ii. alt. If the eigenvalues are not real determine the directions of the prinicipal axes of the associated elliptical orbits. Determine the direction of rotation.

1. (6 + 4 points).

$$X' = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix} X .$$

Solution. i. Call the matrix A. Then

$$\det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & -4 \\ 4 & -5-\lambda \end{pmatrix} = \lambda^2 - 25 + 16 = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3).$$

There are two distinct eigenvalues 3 and -3.

Compute the eigenvector for $\lambda = 3$. One has

$$A - 3I = \begin{pmatrix} 5-3 & -4 \\ 4 & -5-3 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 4 & -8 \end{pmatrix} .$$

$$(x_1, x_2) \in \ker \begin{pmatrix} 2 & -4 \\ 4 & -8 \end{pmatrix} \quad \Leftrightarrow \quad x_1 - 2x_2 = 0 .$$

The kernel consists of vectors $(2x_2, x_2) = x_2(2,1)$*. Those with* $x_2 \neq 0$ *are the eigenvectors.* $e^{3t}(2,1)$ *is a solution.*

Prob.	Score	Out of
1		10
2		10
3		10
4		10
5		10
Sum, %		

For $\lambda = -3$,

$$A + 3I = \begin{pmatrix} 8 & -4 \\ 4 & -2 \end{pmatrix}.$$

The vector (x_1, x_2) is in the nullspace if and only if $2x_1 - x_2 = 0$. *The kernel consists of vectors* $(x_1, 2x_1) = x_1(1, 2))$. *Those with* $x_1 \neq 0$ *are the eigenvectors.* $e^{-3t}(1, 2)$ *is a solution.*

The general solution is

$$a\, e^{3t}(2, 1) + b\, e^{-3t}(1, 2)$$

with arbitrary scalars a *and* b.

ii. The line $(1, 2)\mathbb{R}$ is invariant. On this line solutions approach the origin exponentially as $t \to \infty$. It is the stable manifold.

The line $(2, 1)\mathbb{R}$ is the other invariant line. Solutions on this line grow exponentially as $t \to \infty$. It is the unstable manifold.

The orbits are hyperbolas since the eigenvalues are of opposite sign and equal. The orbits approach the line through $(2, 1)$ as t approaches infinity and approach the line through $(1, 2)$ as t approaches $-\infty$.

There is a nonconstant continuos conserved quantity but you were not asked about it.

2. (5+5 points).

$$X' \;=\; \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} X$$

Solution. Call the matrix A. Then

$$\det(A-\lambda I) = \det \begin{pmatrix} 1-\lambda & 5 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 - 1 + 5 = \lambda^2 + 4\,.$$

Their are distinct eigenvalues $\lambda - \pm 2i$.

For the eigenvalue $\lambda = 2i$

$$A - \lambda I = A - 2i\, I = \begin{pmatrix} 1-2i & 5 \\ -1 & -1-2i \end{pmatrix}\,.$$

The first row multiplied by $1+2i$ is equal to $5 \;\; 5(1+2i)$ which is -5 times the second row checking that the matrix is singular. (If you compute as poorly as I do it is good to have verifications like this. I also use Matlab to check my arithmetic.)

The vector (x_1, x_2) belongs to the kernel if and only if

$$(1-2i)x_1 + 5x_2 = 0, \qquad x_2 = \frac{-1+2i}{5} x_1, \qquad (x_1,x_2) = x_1\Big(1, \frac{-1+2i}{5}\Big)\,.$$

$e^{2it}\Big(1, \frac{-1+2i}{5}\Big)$ *is a solution. It's complex conjugate is another. The general solution is*

$$a\, e^{2it}\Big(1, \frac{-1+2i}{5}\Big) \;+\; b\, e^{-2it}\Big(1, \frac{-1-2i}{5}\Big)$$

with a, b *arbitrary complex numbers.*

ii. alt. The eigenvalues being a pair of complex conjugate purely imaginary numbers, the orbits are ellipses. To find the direction of rotation compute the tangent vector to the orbit through the point $(1,0)$. It is equal to the product of A times $(1,0)$ which is the first column of A, hence $(1,-1)$. *This points downward from the point* $(1,0)$ *so the direction of rotation on the ellipse is clockwise.*

The directions X of the principal axes are the vectors for which X and the direction of motion AX are orthogonal. This yields the condition

$$0 = X\cdot AX = (x_1,x_2)\cdot(x_1+5x_2, -x_1-x_2) = x_1(x_1+5x_2) + x_2(-x_1-x_2) = x_1^2 + 4x_1x_2 - x_2^2\,.$$

Nonzero solutions must satisfy $x_2 \neq 0$, and then $y := x_1/x_2$ satisfies $y^2+4y-1=0$. This has the two roots

$$y_\pm := \frac{-4 \pm \sqrt{16-4(-1)}}{2} = \frac{-4\pm\sqrt{20}}{2} = -2\pm\sqrt{5}\,.$$

Taking $x_2 = 1$ *yields the directions of the axes* $(-2\pm\sqrt{5}, 1)$.

Since $y_+y_- = -1$ these directions are orthogonal yielding a check on the arithmetic.

Since you are not asked to compute the relative lengths of the principal axes you cannot sketch the ellipses. I did not ask for the relative lengths for two reasons. First to keep the exam short. And second the formulas get a little ugly. You should however know how to do this from the handout and homework.

There is a conserved quantity which is a quadratic polynomial in (x_1, x_2). I did not ask you about it but might have.

3. (3+5+2 points). For small $0 < \epsilon << 1$ consider the slightly nonlinear initial value problem,

$$x' \;=\; t \;+\; \epsilon\, x^{10}\,, \qquad x(0) \;=\; 0\,.$$

i. Determine the unperturbed solution.

ii. Find an initial value problem determining the order ϵ corrector from perturbation theory.

iii. Solve that initial value problem to find an approximate solution.

Solution. i. Denote by $x(t,\epsilon)$ the solution. The unperturbed solution $x(t,0)$ is the solution of the initial value problem obtained when $\epsilon = 0$,

$$x' \;=\; t\,, \qquad x(0) \;=\; 0\,.$$

The Fundamental Theorem of Calculus yields

$$x(t) = x(0) + \int_0^t dx/dt\; dt = 0 + \int_0^t t\; dt = t^2/2\,.$$

ii. Compute first a differential equation satisfied by $\partial x/\partial\epsilon$ for all t, ϵ. The Fundamental Existence Theorem implies that x is an infinitely differentiable function of t, ϵ. Differentiating with respect to ϵ the equation satisfied by $x(t,\epsilon)$ yields

$$\frac{\partial}{\partial t}\frac{\partial x}{\partial \epsilon} = 0 + x^{10} + \epsilon\, 10\, x^9\, \frac{\partial x}{\partial \epsilon}\,.$$

Define

$$z(t) := \frac{\partial x}{\partial \epsilon}\Big|_{\epsilon=0}\,.$$

Setting $\epsilon = 0$ in the preceding equation yields

$$z' = x(t,0)^{10} = (t^2/2)^{10} = t^{20}/2^{10}\,. \qquad (1)$$

Differentiating the relation $x(0,\epsilon) = 0$ with respect to ϵ shows that $\partial x(0,\epsilon)/\partial\epsilon = 0$. Therefore,

$$z(0) = 0\,. \qquad (2)$$

The equations (1), (2) constitute an initial value problem determining z.

iii. The initial value problem for z is solved using the Fundamental Theorem of Calculus,

$$z(t) = z(0) + \int_0^t dz/dt\; dt = 0 + \int_0^t t^{20}/2^{10}\; dt = \frac{t^{21}}{21}\,\frac{1}{2^{10}}\,.$$

The approximate solution is

$$x(t,\epsilon) \approx x(t,0) + \epsilon\,\frac{\partial x(t,0)}{\partial \epsilon} = \frac{t^2}{2} + \epsilon\,\frac{t^{21}}{21}\,\frac{1}{2^{10}}\,.$$

4. (2+3+2+3 points). Consider the scalar ordinary differential equation

$$x' \;=\; -x^3 \;+\; 8\,\sin t \;:=\; f(t,x)\,. \tag{1}$$

i. Show that for $x > 2$ one has $f(t,x) < 0$.

ii. Show that if $x(0) > 2$ then the solution of equation (1) satisfies $x(t) < x(0)$ for $t > 0$. **Hint.** Draw the direction field at the points $(t, x(0))$, $-\infty < t < \infty$. Similarly if $x(0) < -2$ then $x(t) > x(0)$ for $t > 0$ (no demonstration required).

iii. The function f is 2π periodic in time. Denote by $p(x)$ the Poincaré map advancing time by 2π units. Use the results of **ii.** to conclude that $p(x) < x$ for $x > 2$ and $p(x) > x$ for $x < -2$.

iv. Conclude that there is a periodic orbit with $-2 \le x(0) \le 2$.

Advice. Even if you do not get one of the questions, you are allowed to use the result of that question to address the next question.

Solution. This problem is very similar to the second problem on Homework 2.

i. If $x > 2$ then $x^3 > 8$ and $-x^3 < -8$ so

$$f(t,x) = -x^3 + 8\sin t < -8 + 8\sin t < 0$$

since $\sin t \le 1$.

ii. Consider the direction field of the ordinary differential equation in the (t,x) plane. The tangent to the solution curve $(t, x(t))$ is the vector $(1, x'(t)) = (1, -x^3 + 8\sin t)$.

Consider the halfspace $\mathbf{H} := \{(t,x) \;:\; x \le x(0) \quad t \in \mathbb{R}\}$. Its boundary is the horizontal line $\{(t, x(0)) \;:\; t \in \mathbb{R}\}$. Along this line, the second component of this vector is equal to

$$-x(0)^3 + 8\sin t < 0$$

by the result of part **i.** This shows that the direction field points into **H** at all points of its boundary.

It follows *that solutions with initial value $x(0)$ on this boundary lie strictly inside the half space for all $t > 0$* since for small time they move inside and there is no way for them to return to the boundary since the direction field is opposed on the boundary.

The phrase in italics is the desired conclusion.

A rigorous argument considers the first time $t_1 > 0$ when a solution returns to the boundary of **H** from below. Since the solution has $x(t_1) = 2$ and $x(t) < 2$ for $t < t_1$ one must have $x' \ge 0$. Part **i** shows $x' < 0$ a contradiction.

Alternate ii. Suppose that $\underline{t} > 0$ and $x(\underline{t}) \ge x(0)$. Consider $x(t)$ as the solution of the initial value problem with initial time $\underline{t}$.

Consider the half space $\mathbf{K} := \{(t,x) \;:\; x \ge x(\underline{t})\}$. The direction field points downward (and therefore out of **K**) on the boundary of **K**. It follow that for $t < \underline{t}$ the solution lies in K. From part **i**, $x' < 0$ so for $0 \le t \le \underline{t}$, $x(t)$ is strictly decreasing so $x(\underline{t}) < x(0)$ contradicting the supposition.

iii. By definition, $p(\underline{x})$ is the value at time 2π of the solution with $x(0) = \underline{x}$. The result of part **ii** shows that if $\underline{x} > 2$ then $p(\underline{x}) < \underline{x}$. The assertion of **ii** after the hint shows that if $\underline{x} < -2$ then $p(\underline{x}) > \underline{x}$.

iv. The Fundamental Existence Theorem implies that p is infinitely differentiable so continuous. Let $\underline{x}$ decrease to 2. Since $p(\underline{x}) < \underline{x}$, the continuity of $p(x)$ implies $p(2) \leq 2$. Similarly letting $\underline{x}$ increase to -2 shows that $p(-2) \geq -2$. (It is not hard to show $p(2) < 2$ and $p(-2) > -2$ but that is not needed.)

Consider the continuous function $x \mapsto g(x) := p(x) - x$. We have shown that $g(2) \leq 0$ and $g(-2) \geq 0$. The intermediate value theorem implies that there is a $c \in [-2, 2]$ so that $g(c) = 0$. Therefore $p(c) = c$. Thus the solution with initial value $x(0) = c$ is 2π periodic.

5. (3+3+4 points). Consider

$$X' \;=\; \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix} X \;:=\; A\,X\,, \qquad \text{satisfying,} \qquad \det(zI - A) \;=\; (z+1)^2(z-1)\,.$$

In addition to the determinant on the right, you are given the information that for the eigenvalue 1, $(0,2,1)$ is an eigenvector.

i. Find all eigenvector(s) for the eigenvalue -1. †

ii. Find a basis for the generalized eigenpace corresponding to the eigenvalue -1.

iii. Find the general solution of the differential equation.

Solution. i. For the eigenvalue $\lambda = -1$ one has

$$A - \lambda I = A + I = \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}\,.$$

The last two rows are proportional. The kernel of this matrix is the set of vectors (x_1, x_2, x_3) that satisfy

$$x_2 - 2x_3 = 0, \qquad \text{and} \qquad 2x_1 + x_2 = 0\,.$$

Solving for x_1 and x_3 in terms of x_2 yields the vectors $x_2(-1/2, 1, 1/2)$. The kernel has dimension 1. It is spanned by $(-1,2,1)$. The eigenvectors are the nonzero multiples of this vector.

ii. The root $\lambda = -1$ of $\det(zI - A) = 0$ has multiplicity 2. Therefore the generalized eigenspace is the kernel of $(A - \lambda)^2 = (A+I)^2$. Compute

$$(A+I)^2 = \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 8 & -8 \\ 4 & 4 & -4 \end{pmatrix}\,.$$

The kernel is the set of vectors (x_1, x_2, x_3) that satisfy $x_1 + x_2 - x_3 = 0$. Solving for x_1 in terms of x_2 and x_3 yields

$$(x_3 - x_2\,,\, x_2, x_3) = x_2(-1,1,0) + x_3((1,0,1)$$

which produces the basis vectors $(-1,1,0)$ and $(1,0,1)$ for the generalized eigenspace associated to eigenvalue -1.

Since the multiplicity is two we know that the generalized eigenspace will have dimension two. This is a check on the arithmetic.

iii. The solutions associated to the eigenvalue $\lambda = 1$ are constant multiples of $Y_1(t) := e^t(0,2,1)$.

We need to construct two linearly independent solutions associated to the eigenvalue $\lambda = -1$. Section **3** of the Spectral Decomposition Handout shows that if v is a vector in the generalized eigenspace corresponding to an eigenvalue λ with multiplicity μ, then

$$e^{\lambda t}\Big(I + \frac{(A-\lambda I)t}{1!} + \cdots + \frac{(A-\lambda I)^{\mu-1}t^{\mu-1}}{(\mu-1)!}\Big)v$$

† You are reminded that by definition eigenvectors are nonzero vectors.

is a solution.

Taking $\lambda = -1$, $\mu = 2$ and $v = (-1, 1, 0)$ yields the solution

$$Y_2(t) := e^{-t}\left(I \; + \; \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} t\right)(-1, 1, 0)\,. \tag{3}$$

Taking $v = (1, 0, 1)$ yields the solution

$$Y_3(t) \; := \; e^{-t}\left(I \; + \; \begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} t\right)(1, 0, 1)\,. \tag{4}$$

The general solution is

$$a\, Y_1(t) \; + \; b\, Y_2(t) \; + \; c\, Y_3(t) \tag{5}$$

with arbitrary scalars a, b, and c.

Different basis vectors yield different formulas (5). It is only in the unsimplified forms of (3) and (4) that the answers are easily checked.