Math 558, Fall 2011 Prof. J. Rauch

Solutions to Final Exam December 17, 2011

Instructions. 1. Two 3.5in. × 5in. sheet of notes from home. Four sides. Closed book.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are 6 questions, 6 pages, and a total of 52 points.
5. You may use the back of the pages and/or supplementary sheets.

1. (3+3+3 points). Consider

$$X' = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{pmatrix} X := A\,X\,.$$

You are given the information that the matrix satisfies

$$\det(zI - A) = (z+1)^2(z+2)\,.$$

and for the eigenvalue -2, $(1,-1,0)$ is an eigenvector.

i. Find all eigenvectors for the eigenvalue -1. †

ii. Find a basis for the generalized eigenspace corresponding to the eigenvalue -1.

iii. Find the general solution of the differential equation.

Solution. i. For the eigenvalue -1 compute

$$A - (-1)I = A + I = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

The eigenvectors are the nonzero elements of the kernel. Row reduction shows that these are exactly the vectors $x_2(-1,1,-1)$ with $x_2 \neq 0$.

ii. The space of eigenvectors has dimension 1 and the root of the characteristic polynomial has multiplicity 2. The generalized eigenspace is then the nullspace of

$$(A+I)^2 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nullspace consists of vectors satisfying the single equation $x_2 = -x_3$. It has dimension 2. It is spanned by the eigenvector and any second vector in the nullspace that is not parallel to the eigenvector, for example $(0,1,-1)$.

There is a good check on the arithmetic at this stage. The eigenvector found in **i** must be in the generalized eigenspace. If you make an arithmetic error you usually find a nullspace for $(A-(-1)I))^2$ that is neither two dimensional nor contains this eigenvector.

† By definition eigenvectors are nonzero vectors.

iii. The eigenvectors yield the independent solutions

$$X_1(t) \;:=\; e^{-2t}(1,-1,0)\,, \qquad X_2(t) \;:=\; e^{-t}(1,-1,1)\,.$$

A third solution is

$$X_3(t) \;:=\; e^{-t}\Big(I + t\big(A-(-1)I\big)\Big)(0,1,-1) \;=\; e^{-t}\left(I + t\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The general solution is a linear combination

$$c_1X_1(t) \;+\; c_2X_2(t) \;+\; c_3X_3(t)\,.$$

A common error was to include a term $t^2(A{+}I)^2/2$. However this matrix annihilates the generalized eigenspace so adds zero. It is not wrong but demonstrates a lack of understanding of why the and where the exponential series terminates.

2. (3+4 points). Consider the differential equation

$$x' \;=\; x^4 \;-1 \;+\; a \;:=\; f(a,x)$$

with real parameter a.

i. For all parameter values determine the equilibria. In particular find all bifurcation points.

ii. For all a determine the stability of each equilibrium.

Solution i. The equilibria are solutions of $x^4 = 1-a$. There are no solutions for $a>1$, exactly one solution $x=0$ when $a=1$, and two solutions $x=\pm(1-a)^{1/4}$ for $a<1$. There is a bifurcation at $a=1$.

ii. The next arguments are better done with sketch of the graphs and a phase line diagram. The written solution is best appreciated by drawing the graph.

For $a=1$ the equation is $x'=x^4$. The orbit direction is to the right on both sides of the equilibrium $x=0$. Orbits starting to the right of $x=0$ are monotonically increasing and for any $\underline{x}$ pass the point $\underline{x}$ at finite time. Otherwise it would stay to the left and $\lim_{t\to\infty} x(t)$ would be an equilibrium in $]x(0),\underline{x}]$. No such equilibrium exists. So the equilibrium is unstable.

The stability for $a<1$ is verified either by the phase line diagram with $f>0$ to the left of $-(1-a)^{1/4}$ negative on the interval between the equilibria, and positive to the right of $(1-a)^{1/4}$. Thus the left hand equilibrium is asymptotically stable and the right hand equlibrium unstable and attracts solutions as $t\to-\infty$.

This stability can also be checked by the deivative test. That $f_x<0$ at the left hand equlibrium and $f_x>0$ at the right hand is clear from the graph. An analytic proof computes

$$f_x \;=\; 4x^3\,, \qquad \mathrm{sign}\, f_x \;=\; \mathrm{sign}\, x\,.$$

3. (3+2+2 points). For the system

$$\dot{x} \;=\; \ln(1+y+x^2)\,, \qquad \dot{y} \;=\; 4x + xy\,,$$

the origin is an equilibrium.

i. Compute the linearization at the origin.

ii. Show that the linearization is a saddle.

iii. Determine the tangent direction to the unstable manifold of the nonlinear system at the origin.

Solution. The fast linearization is by Taylor expansion for small x, y using the fact that the derivative of $\ln(s)$ at $s = 1$ is one to find

$$\dot{x} \approx y + x^2 \approx y\,, \qquad \dot{y} \approx 4x + xy \approx 4x$$

to find the linearization

$$\dot{X} \;=\; \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} X \;:=\; AX\,. \tag{3.1}$$

The slow way is to set $u(x, y) = \ln(1 + y + x^2)$ and compute

$$\frac{\partial u}{\partial x} \;=\; \frac{1}{1 + y + x^2}\, 2x\,, \qquad \frac{\partial u}{\partial y} \;=\; \frac{1}{1 + y + x^2}\,.$$

Setting $x = y = 0$ yields the values 0 and 1, the first row of the matrix in (3.1). The second row consists of the partial derivative of $v(x, y) = 4x + xy$,

$$\frac{\partial v}{\partial x}\bigg|_{0,0} = 4 + y\big|_{0,0} = 4\,, \qquad \frac{\partial v}{\partial y}\bigg|_{0,0} = x\big|_{0,0} = 0\,.$$

ii. To determine the type compute the eigenvalues. They solve

$$0 = \det(zI - A) = \det\begin{pmatrix} z & -1 \\ -4 & z \end{pmatrix} = z^2 - 4\,.$$

Therefore $z = \pm 2$. Real roots of opposite sign. Therefore the linearization is a saddle.

iii. The tangent direction to the unstable manifold is the direction of the eigenvectors of A corresponding to the positive eigenvalue $+2$. These are the nonzero eigenvectors in the kernel of

$$2I - A \;=\; \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

The rows are proportional a good arithmetic check. The nullspace consists of vectors satisfying $2x_1 = x_2$. The tangent direction is $(1, 2)$. The stable manifold has slope equal to 2 at the origin.

4. (4+2+4 points). **i.** For the system

$$X' \;=\; \begin{pmatrix} -1 & 10 \\ 0 & -1 \end{pmatrix} X$$

show that for Λ a sufficiently large postive constant, the function $L(X) := x_1^2 + \Lambda\, x_2^2$ is a strict Lyapunov function on the whole plane. **Hint.** Some demonstations make clever use of the inequality $2ab \le a^2 + b^2$.

ii. Show that the same function is a strict Lyapunov function on $\mathbb{R}^2$ for the nonlinear system

$$X' \;=\; (x_1^2 + x_2^2) \begin{pmatrix} -1 & 10 \\ 0 & -1 \end{pmatrix} X \;:=\; F(X)\,. \tag{4.1}$$

iii. Show that **all solutions** of (4.1) converge to the origin as $t \to \infty$. **Hint.** Use Lasalle's principal. **Discussion.** The stability of the equilibrium cannot be determined by linearization.

Solution. i. If $\Lambda > 0$ then L has a strict minimum at the origin where $L = 0$.

Compute

$$\dot{L} \;=\; \nabla_x L \cdot F(X) \;=\; \Big(2x_1\,,\, 2x_2\Big) \cdot \Big(-x_1 + 10x_2\,,\, -x_2\Big) \;=\; -2\Big(x_1^2 + 10x_1x_2 + \Lambda x_2^2\Big)\,.$$

To show that L is a strict Lyapunov function is equivalent to showing that the quadratic form in parentheses is strictly positive definite.

Since the form is positive on the axes for $\Lambda > 0$ it is sufficient to show that it cannot vanish. The absence of roots is equivalent to the discriminant condition $10^2 - 4\Lambda < 0$. The form is positive definite if and only if $\Lambda > 25$.

The same sharp result is obtained if one estimates for any $0 < a < 1$

$$10x_1x_2 = 2(ax_1)(5/a) \le a^2x_1^2 + (5/a)^2x_2^2\,,$$

to find

$$x_1^2 - 10x_1x_2 + \Lambda x_2^2 \ge x_1^2 - a^2x_1^2 - (5/a)^2x_2^2 + \Lambda x_2^2 = (1-a^2)x_1^2 + (\lambda - 25/a^2)x_2^2\,.$$

Choosing a close to 1, this shows that the form is positive definite for any $\Lambda > 25$.

A third argument completes the square

$$x_1^2 + 10x_1x_2 = (x_1 - 5x_2)^2 - 25x_2^2\,.$$

Thus

$$\dot{L} \;=\; (x_1 - 5x_2)^2 - (\Lambda - 25)x_2^2\,.$$

The form is positive definite for $\Lambda > 25$.

ii. Compute

$$\dot{L} \;=\; (x_1^2 + x_2^2)(-2)(x_1^2 - 10x_1x_2 + \Lambda x_2^2)\,.$$

That L is a strict Lyapunov function for $\Lambda > 25$ follows from **i**.

iii. For any $C > 0$ the set $\mathcal{P}$ of X so that $L(X) \le C$ is closed bounded and positive invariant. Since $\dot{L} < 0$ on $\mathcal{P} \setminus 0$ the only orbit in $\mathcal{P}$ on which $L = \textit{constant}$ is the equilibrium. If follows from LaSalle's Invariance Principal that all orbits in $\mathcal{P}$ tend to the origin.

Since the sets $\mathcal{P}$ increase to $\mathbb{R}^2$ as C increases to ∞ it follows that all orbits tend to the origin.

A common error was to fail to show that orbits lie in a closed bounded set. Or to construct a $\mathcal{P}$ that is not bounded. Or a $\mathcal{P}$ that is not positive invariant.

5. (5+5 points). Hirsch-Smale-Devaney 213/8c variant. Consider the system

$$x' \;=\; y^2 \;+\; 2xy\,, \qquad y' \;=\; x^2 \;+\; 2xy\,.$$

i. Find the unique equilibrium.

ii. Show that this is a gradient system and find $V(x,y)$ with $V(0,0)=0$ (show work). **Discussion.** Without $V(0,0)=0$, V would only be determined up to a constant.

iii. Show that the diagonal $\{x=y\}$ is invariant under the flow.

iv. Sketch the flow on the diagonal. In particular, show that the equilibrium is unstable.

Solution. i. At an equilibrium must have $x'=y'=0$. The first equation yields $y(y+2x)=0$ so $y=0$ for $y=-2x$

Plug $y=0$ into $y'=0$ to find $x^2=0$ whence $x=0$. Similarly plugging $y=2x$ into $y'=0$ yields

$$0 = x^2 + 2x(-2x) = -3x^2\,.$$

Again $x=0$. Thus $(0,0)$ is the only equilibrium.

ii. With $f := y^2+2xy$ and $g := y^2+2xy$ the system is gradient if and only if $f_y = g_x$. Compute

$$f_y = 2y+2x\,, \qquad g_x = 2y+2x\,,$$

so the system is gradient.

Need $\partial V/\partial x = y^2+2xy$. Thus $V = xy^2 + x^2y + h(y)$. Then must have

$$x^2+2xy = V_y = x^2+2xy+h'(y)\,.$$

Therefore $h'=0$ so $h=C$ a constant. The most general potential is xy^2+yx^2+C. The unique potential with $V(0,0)=0$ is $V(x,y)=xy^2+yx^2$.

iii. Let $K(t) := x(t)-y(t)$. Then on orbits

$$\frac{d}{dt}K = x'-y' = y^2-x^2 = (x+y)K\,.$$

Define $A(t) := y(t)+x(t)$. For the orbit with $x(t_0)=y(t_0)$ have

$$K' = A(t)K\,, \qquad K(t_0)=0\,.$$

A solution of this initial value problem is $K=0$. By uniqueness it is the only one. Therefore $K=0$ so $x(t)-y(t)$ is identically zero showing that the orbit lies on the diagonal.

A geometric argument is that for $x=y$ the tangent vector is

$$(x',y') \;=\; (y^2+2xy\,,\,x^2+2xy) \;=\; 3x^2(1,1)\,.$$

On the diagonal the velocity is tangent to the diagonal.

Analytically, for an initial point (x_0, x_0) on the diagonal the function $(x(t), x(t))$ is a solution of the system if and only if

$$x' = 3x^2 \,. \tag{5.1}$$

The phase line for this equation has arrows to the right and a unique equilibrium at the origin. Solutions starting with $x > 0$ grow infinitely large. In fact in finite time. They can be found exactly by separation of variables in (5.1)

6. (3+3+3 points). Consider the family of **maps** $f(x) = (x-a)^2$ from the real line $\mathbb{R}$ to itself. **Warning.** Maps not differential equations!

i. Show that for $a < -1/4$ there are no fixed points, for $a = -1/4$ one fixed point and for $a > -1/4$ two fixed points.

ii. Explain why the left hand equilibrium is stable for a slightly larger than $-1/4$. And the right hand equilibrium is unstable. **Hint.** Consider graphs. Use derivative test for stability.

As a increases it is true that the slope of f at the left hand fixed point decreases. For $a = 0$ the slope is zero (graph!) and for larger values of a the slope becomes more and more negative reaching the slope -1 when $a = 3/4$. *You need not verify these statements.*

iii. Describe the bifurcation that occurs at $a = 3/4$.

Solution. i. Fixed points are solutions of $(x-a)^2 = x$ a quadratic equation. Simplifying yields

$$x^2 - (2a+1)x + a^2 = 0\,, \qquad \text{discriminant} = (2a+1)^2 - 4a^2 = 4a+1\,.$$

When $a < -1/4$ the discriminant is negative and there are no roots. Geometrically, the parabola $y = (x-a)^2$ lies strictly above the line $y = x$.

When $a = -1/4$ there is one double root at $x = 1/4$. At that value of x the graph of $y = (x-(-1/4))^2$ and $y = x$ touch at a point of tangency.

For larger values of a the parabola moves right and intersects at two fixed points

$$\frac{(2a+1) \pm \sqrt{4a+1}}{2}\,.$$

An interesting special case is $a = 0$ for which the two fixed points are $x = 0$ and $x = 1$.

ii. This result is most easily seen geometrically. For a just larger than $-1/4$ the parabola just passes to the right of the line $y = x$. At the left hand fixed point, the parabola has slope slightly less than the slope of the line and at the right hand fixed point the slope is slightly greater than one. The derivative test yields the stabilties requested.

The same can be verified algebraically. For a slightly larger than $1/4$ the quantity $\epsilon = \sqrt{4a+1}$ is small positive. The fixed points are

$$\frac{(2a+1) \pm \epsilon}{2} \;=\; a + \frac{1 \pm \epsilon}{2}\,. \tag{6.1}$$

For the derivative test need to compute $f'(x) = 2(x - a)$ at the fixed points. Using (6.1) one has

$$f'(x_{\pm}) \;=\; 1 \pm \epsilon\,.$$

The left hand fixed point is x_- and the derivative is $1 - \epsilon < 1$ yielding stability. For the right hand root $f'(x_+) = 1 + \epsilon > 1$ demonstrating instability.

iii. We are given that $f'(x_-)$ is negative and decreasing with a passing through -1 when $a = 3/4$. At that value the left hand fixed point changes from stable to unstable.

Considering the fixed point $x_-(a)$ as a fixed point of $f \circ f$, the derivative of $f \circ f$ passes from less than one through the value one and then to larger values. This is exactly the situation of bifurcation. One has the fixed point x_- of $f \circ f$ for $a < 3/4$, one for $a = 3/4$ and for $a > 3/4$ this fixed point plus two more that form a stable 2-cycle for f bouncing from one side of x_- to the other. The stable fixed point x_- loses its stability to the stable two cycle. This is the classic *period doubling bifurcation.*

For values of a slightly larger than 3/4 the fixed points of $f \circ f$ (including $x_{\pm}$) satisfy a quartic equation. The 2-cycle can be found exactly by factoring the quartic using the fact that we know the two roots $x_{\pm}$.