Math 558, Fall 2011 Prof. J. Rauch

Midterm Exam, October 20, 2011

Instructions. 1. Two sides of a 3.5in. × 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, . . . etc.
3. Show work and explain clearly.
4. There are six questions. Each is worth 10 points.
5. You may use the back of the pages.

1. (10=2+4+3+1 points). **i.** For the differential equation

$$X' \;=\; \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} X\,,$$

show that the phase portrait is a saddle.

ii. Find the general solution.

iii. Sketch the phase portrait indicating all invariant lines and their stability and typical orbits.

iv. Determine if the orbits are hyperbolas.

Solution. i. Compute

$$\det(\lambda I - A) \;=\; \det\begin{pmatrix} \lambda+2 & -1 \\ 4 & \lambda-3 \end{pmatrix} \;=\; (\lambda+2)(\lambda-3)+4 \;=\; \lambda^2-\lambda-2 \;=\; (\lambda-2)(\lambda+1)\,.$$

The eigenvalues are $\lambda = 2, -1$ are real and of opposite sign so the phase portrait is a saddle.

ii. Use the eigenvalue– eigenvector method. For $\lambda = 2$,

$$\lambda I - A \;=\; \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}.$$

The fact that the matrix is clearly singular is a good check on the computation of the eigenvalues. The eigenvectors are the nonzero vectors (x_1, x_2) that satisfy $x_2 = 4x_1$. These are the multiples of $(1, 4)$. The associated solutions of the DE are the constant multiples of $e^{2t}\,(1, 4)$.

For $\lambda = -1$,

$$\lambda I - A \;=\; \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}.$$

The eigenvectors satisfy $x_1 = x_2$ and the associated solutions are constant multiples of $e^{-t}(1, 1)$. The general solution is

$$c_1\, e^{2t}\,(1, 4) \;+\; c_2\, e^{-t}(1, 1)\,.$$

iii. The line with slope one is the stable manifold while the line with slope 4 is the unstable manifold. Orbits are asymptotic to the latter as $t \to \infty$ and to the former when $t \to -\infty$.

iv. The orbits of a saddle are hyperbolas if and only if the eigenvalues of opposite sign have equal magnitude. This is not the case for the present eigenvalues, $|-1| \neq |2|$.

2. (10=3+2+2+3 points). **i.** For the differential equation

$$X' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} X,$$

show that the phase portrait is of spiral type.

ii. Determine the stability of the equilibrium $(0,0)$.

iii. Determine the direction of rotation about the equilibrium.

iv. Determine the prinicipal axes of the associated elliptical orbits of $X' = (A - ((\mathrm{tr}\, A)/2))I)X$.

Solution. The characteristic polynomial is

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+3 & -2 \\ 1 & \lambda+1 \end{pmatrix} = (\lambda+3)(\lambda+1)+2 = \lambda^2+4\lambda+5\,.$$

The eigenvalues are

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{4}}{2} = -2 \pm i\,.$$

The eigenvalues are a complex conjugate pair with nonzero real and imaginary parts so the phase portrait is a spiral.

ii. The real parts are negative so the phase portrait is a spiral attractor and the equilibrium is stable.

iii. Compute the direction field at the point $X = (1,0)$,

$$\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-3,-1)\,.$$

Since the x_2-component is negative it follows that the rotation is clockwise. Draw a sketch.

iv. Compute

$$\widetilde{A} := A - \frac{\mathrm{tr}\, A}{2} I = A + 2I = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}.$$

The directions of the ellipse axes satisfy

$$0 = \widetilde{A}X \cdot X = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X \cdot X = (-x_1+2x_2, -x_1+x_2)\cdot(x_1,x_2) = -x_1^2 + x_1x_2 + x_2^2\,.$$

The only solution with $x_1 = 0$ is $(0,0)$. Dividing by x_1^2 shows that for any non zero solution $y := x_2/x_1$ satisfies

$$y^2+y-1=0\,, \qquad y = \frac{-1 \pm \sqrt{1-4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}\,.$$

Taking $x_1 = 1$ shows that the axes of the ellipse point in the directions

$$\left(1\,, \frac{-1+\sqrt{5}}{2}\right), \qquad \text{and}\,, \qquad \left(1\,, \frac{-1-\sqrt{5}}{2}\right).$$

3. (10=3+7 points). For small $0 < \epsilon << 1$ consider the slightly nonlinear initial value problem,

$$(1 + \epsilon\, e^x)\, x' \;=\; x, \qquad x(0, \epsilon) \;=\; 1$$

determining $x(t, \epsilon)$.

i. Find the unperturbed solution $x(t, 0)$.

ii. Find an initial value problem determining the order ϵ corrector term, $z(t)$ in the perturbation theory approximation

$$x(t, \epsilon) \;\approx\; x(t, 0) \;+\; \epsilon\, z(t) \;+\; \text{higher order terms}\,.$$

You need not solve the initial value problem for $z(t)$.

Solution. Set $\epsilon = 0$ to determine $x(t, 0)$. Find

$$x'(t, 0) = x(t, 0)\,, \qquad x(0, 0) = 1\,,$$

with solution $x(t, 0) \;=\; e^t$.

ii. Differentiating the equation satisfied by $x(t, \epsilon)$ with respect to ϵ yields an equation satisfied by $\partial x/\partial \epsilon$ for all t, ϵ,

$$\big(1 + \epsilon e^x\big)\bigg(\frac{\partial x}{\partial \epsilon}\bigg)' + e^x\, x' + \epsilon\, e^x\, \frac{\partial x}{\partial \epsilon}\, x' \;=\; \frac{\partial x}{\partial \epsilon}\,.$$

Setting $\epsilon = 0$ yields a differential equation satisifed by $z(t) := \partial x(t, 0)\partial \epsilon$,

$$z' \;+\; e^x\, x' \;=\; z\,.$$

At $t = 0$, $x' = x = e^t$ so

$$z' \;+\; e^{e^t} e^t \;=\; z\,. \tag{1}$$

Differentiating the initial condition $x(0, \epsilon) = 1$ with respect to ϵ yields

$$z(0) \;=\; 0\,. \tag{2}$$

Equations (1) and (2) are the desired initital value problem for z. One has the first order Taylor approximation

$$x(t, \epsilon) \;\approx\; e^t \;+\; \epsilon\, z(t)\,.$$

4. (10=5+2+3 points). Consider the scalar ordinary differential equation

$$x' \;=\; (x+3)(x-1)(x-2)^2 \;:=\; f(x)\,. \tag{1}$$

i. Find all equilibria. Determine the stability of each equilibrium.

ii. For the solution with $x(0) = 0.5$ determine $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.

iii. For each equilibrium, determine whether the equilibrium is structurally stable, that is persists with similar local phase portrait under small perturbation of f.

Solution. **i.** The equilibria are the zeroes of the function $f(x)$ namely $x = -3, 1, 2$. On the interval $-\infty, -3[$, f is positive crossing to negative on $]-3, 1[$ at the equilibrium $x = -3$. Orbits to the left of -3 increase and those in $]-3, 1[$ decrease. The equilibrium -3 is stable.

The fundamental theorem of the phase line implies that solutions starting just to the left of 1 decrease to -3 as $t \to \infty$ proving the instability of the equilibrium 1.

The function f is postive on both sides of the equilibrium $x = 2$. Orbits starting just to the right of $x = 2$ increase diverging to $x = \infty$. In fact they diverge to infinity in finite time, though that is not needed. One concludes that the equilibrium $x = 2$ is unstable.

ii. In the interval $]-3, 1[$ the function f is stricIty negative. The fundamental theorem of the phase line implies that that solutions starting in $]-3, 1[$ decrease to -3 as $t \to \infty$. This holds inparticular for $x(0) = 0.5$. In the same way as t decreases to $-\infty$ the orbit increases to the nearest equilibrium, $x = 1$.

iii. At the equilibria $x = -3$ and $x = 1$ one has $f' \neq 0$. In fact, evaluating f' by the product rule one finds,

$$f' = (x-1)(x-2)^2 \;+\; (x+3)(x-2)^2 \;+\; (x+3)(x-1)2(x-2)\,.$$

Therefore,

$$f'(-3) \;=\; (-3-1)(-3-2)^2\,, \qquad f'(1) \;=\; (1+3)(1-2)^2\,.$$

The graph of $f(x)$ crosses the x-axis transversely at these points. This implies (and is in fact equivalent) to the structural stability of the equilibrium under small perturbations of f.

On the other hand $f'(2) = 0$. If one increases f a lilttle bit near $x = 2$ the equilibrium disappears. If one decreases f the single equilibrium splits into two nearby equilibria. The equilibrium $x = 2$ is not structurally stable.

5. (10=4+4+2 points) Consider the scalar ordinary differential equation

$$x' \; = \; f(t,x)\,,$$

where f is periodic with period 1, that is for all (t,x),

$$f(t+1,x) \; = \; f(t,x)\,.$$

Suppose that it is known that solutions with arbitrary initial values exist for all time and denote the Poincaré map taking $x(0)$ to $x(1)$ by p. Suppose that it s known that $p(x)$ is concave down ($p'' < 0$) and that $p(x) - x$ has roots at $x = 0$ and $x = 1$.

i. How many periodic orbits are there, and what are the values of $x(0)$ for these orbits?

ii. For each periodic orbit determine whether the orbit is stable or not.

iii. For the solution with initial value $x(0) = 0.3$ show that as $t \to \infty$, $x(t)$ converges to a periodic orbit and determine which one.

Solution. **i.** Since $p(x) - x$ is strictly concave down $p(x) - x$ can cross the x-axis at most at two points. This is so since between two crossings there must be a point where the derivative $p\prime$ vanishes. By hypothesis, $(p - x)'$ is strictly monotone decreasing so can vanish at at most one point. Therefore $p(x) - x$ can vanish at most at two. When there are two crossings the derivative is positive at the left hand crossing and negative at the right hand crossing.

Since $p(x) - x = 0$ at $x = 0$ and at $x = 1$ these are all of its roots. It crosses $x = 0$ with postive slope and $x = 1$ with negative slope. The crossings are the fixed points of $p(x)$ and are therefore the initial data of the periodic orbits. There are exactly two periodic orbits. They have $x(0) = 0$ and $x(0) = 1$.

ii. The strict concavity implies that $p(x) - x < 0$ on $]-\infty, 0[$, $p(x) - x > 0$ on $]0, 1]$, and, $p(x) - x < 0$ on $]1, \infty[$.

The fundamental theorem on monotone maps implies that the orbit $p^n(x)$ decreases towards $-\infty$ for $x < 0$, increases to $x = 1$ for $0 < x < 1$ and decreases toward $x = 1$ for $0 < x < \infty$. It follows in particular that the periodic orbit with $x(0) = 0$ is unstable and that with $x(0) = 1$ is stable.

Alternatively, the stability of the periodic orbit with $x(0) = 1$ follows from the derivative test $p'(1) - 1 < 0$ and the instability of the periodic orbit with $x(0) = 0$ follows from $p'(0) - 1 > 0$.

iii. As noted in **ii**, the orbit $p^n(0.3)$ increases toward $x = 1$. This shows that the orbit with $x(0) = 0.3$ converges to the periodic orbit with $x(0) = 1$.

Discussion. Since the crossings are transverse, this phase portrait of two periodic orbits the lower one unstable and the upper one stable persists under small perturbation of the periodic $f(t,x)$. This includes the possibility of perturbing the period too. The problem is structurally stable. The analysis of the logistic equation with periodic harvesting is an example where the convexity of p could be verified though p itself could not be computed.

6. (10 points). Consider

$$X' \;=\; \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} X \;:=\; A\,X\,, \qquad \text{satisfying,} \qquad \det(zI - A) \;=\; (z-1)(z+2)(z-3)\,.$$

You are given the information that for the eigenvalue 1, $(-1,4,1)$ is an eigenvector and for the eigenvalue -2, $(-1,1,1)$ is an eigenvector.

Find the general solution of the differential equation.

Solution. The characteristic polynomial shows that the 3×3 matrix A has three distinct eigenvalues 1, -2 and 3. Therefore there is a basis of eigenvectors. It suffices to find the eigenvectors.

Eigenvectors for the first two are given in the problem statement. We need an eigenvector for $\lambda = 3$. Compute

$$A - 3I \;=\; \begin{pmatrix} -2, & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}.$$

As the first and last row are proportional this is clearly singular, a good check on the arithmatic. Find the nullspace of $A-3I$ by row reduction. Add last row to first to annihilate it then shift it to the third row to find

$$\begin{pmatrix} 3 & -1 & -1 \\ 2 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppress the last row that represents a trivial equaion. Multiply first row by 2 and second by 3 to find

$$\begin{pmatrix} 6 & -2 & -2 \\ 6 & 3 & -12 \end{pmatrix}.$$

Subtract the second row from first then interchange rows

$$\begin{pmatrix} 0 & -5 & 10 \\ 6 & 3 & -12 \end{pmatrix}, \qquad \text{then,} \qquad \begin{pmatrix} 6 & 3 & -12 \\ 0 & -5 & 10 \end{pmatrix}.$$

Multply first row by $1/3$ and second by $-1/5$

$$\begin{pmatrix} 2 & 1 & -4 \\ 0 & 1 & -2 \end{pmatrix}.$$

Subtract second row from first

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}.$$

Divide first row by 2

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}.$$

The equations now read, $x_1 = x_3$ and $x_2 = 2x_3$. The solutions are parameterized by x_3 as $x_3(1,2,1)$. The eigenvectors are multiples of $(1,2,1)$.

The general solution is

$$c_1\, e^{t}\,(-1,4,1) \;+\; c_2\, e^{-2t}\,(-1,1,1) \;+\; c_3\, e^{3t}\,(1,2,1)$$

the last coming from the computation and the first two from the givens of the problem.