Math 558, Fall 2012 Prof. J. Rauch

Final Exam Solutions   December 18, 2012

Instructions. 1. Two sides of two 3.5in. × 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are 6 questions, 6 pages, and a total of 46 points.
5. You may use the back of the pages and/or supplementary sheets.

1. (3+3 points). (Brauer and Nohel, 61/11d.) The matrix

$$A = \begin{pmatrix} 3 & -1 & -4 & 2 \\ 2 & 3 & -2 & -4 \\ 2 & -1 & -3 & 2 \\ 1 & 2 & -1 & -3 \end{pmatrix}$$

satisfies $\det(A-\lambda I) = (\lambda+1)^2(\lambda-1)^2$ as well as

$$(A+I)^2 = \begin{pmatrix} 8 & 0 & -8 & 0 \\ 8 & 8 & -8 & -8 \\ 4 & 0 & -4 & 0 \\ 4 & 4 & -4 & -4 \end{pmatrix}, \qquad \text{and,} \qquad (A-I)^2 = \begin{pmatrix} -4 & 4 & 8 & -8 \\ 0 & -4 & 0 & 8 \\ -4 & 4 & 8 & -8 \\ 0 & -4 & 0 & 8 \end{pmatrix}.$$

i. Find the generalized eigenspace corresponding to the eigenvalue -1.

ii. Find the general solution of the ordinary differential equation $X' = AX$ with initial data in that generalized eigenspace.

Solution. i. The generalized eigenspace is two dimensional. The set of eigenvectors is one dimensional (computation omitted).

The generalized eigenspace for eigenvalue 1 is the kernel of $(A+I)^2$ since the eigenvalue has multiplicity two. The formula for $(A+I)^2$ shows that the kernel is given by the pair of equations

$$x_1 - x_3 = 0, \qquad x_1 + x_2 - x_3 - x_4 = 0\,.$$

Equaivalently $x_1 = x_3$ and $x_2 = x_4$. Parameterizing by x_3, x_4 the kernel is the set of vectors $x_3(1,0,1,0) + x_4(0,1,0,1)$. Using $e^{tA} = e^{-It}e^{(A+I)t}$, the solutions with data taken from this set are then given by

$$e^{-t}\Big[I + t(A-I)\Big]\Big[a(1,0,1,0) \; + \; b(0,1,0,1)\Big]\,. \tag{1.1}$$

Probl.	1	2	3	4	5	6	Sum	%
Points	6	9	7	9	8	8	47	100

2. (2+4+3 points). (Strogatz 82/3.2.1) Consider the differential equation

$$x' \;=\; ax + 4x^2 \;:=\; f(a,x)\,.$$

i. In the $(a,x)-$plane with the a-axis horizontal, sketch the set of equilibria.

ii. Determine the stability of all equilibria. Indicate the answer by writing "s" or "u" on the stable and unstable branches in the diagram from **i.**

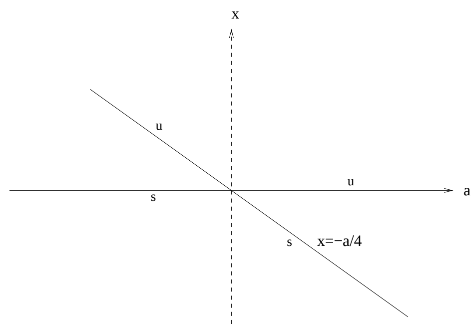
iii. As a varies three qualitatively different phase portraits appear. Sketch the different qualitative behaviors that occur and indicate the regions of a where they occur.

Solution. The equilibria are given by the equation

$$0 = f(a,x) = ax + 4x^2 = x(a+4x)\,.$$

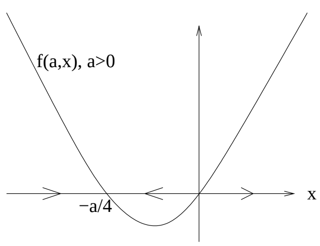
This zero set is the union of the two lines $\{x = 0\}$ and $\{x = -a/4\}$ in the a, x-plane.

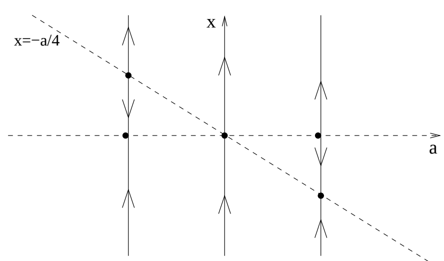
The sketch for $a > 0$ is



The equilibrium is stable when $f_x < 0$ and unstable when $f_x > 0$. Since $f_x = a + 8x$ it follows that the equilibrium $x = 0$ is stable for $a < 0$ and unstable for $a > 0$ as indicated on the sketch.

On the curve of equilibria $a = -4x$, one has $f_x = a + 8x = -4x + 8x = 4x$ so the equilibrium is stable when $x < 0$ and unstable when $x > 0$. The graph of f when $a > 0$ together with the phase line for each of the three cases is sketched below.





3. ((i + ii=3) +4 points). Consider a gradient system

$$x_1' = -V_{x_1}(x_1, x_2), \qquad x_2' = -V_{x_2}(x_1, x_2)$$

with potential V. Suppose that $(\underline{x}_1, \underline{x}_2)$ is an equilibrium.

i. Compute the matrix A of the linearization at $(\underline{x}_1, \underline{x}_2)$,

$$Z' \;=\; A\,Z\,.$$

ii. Show that it is impossible for A to have eigenvalues that are not real.

iii. For

$$V(x,y) = \frac{3x^2}{2} - \frac{3y^2}{2} + 4yx + xy^2$$

show that the origin is an equilibrium that is a saddle in the sense of ordinary differential equations and determine the directions of the stable and unstable manifolds at $(0,0)$.

Solution. i. When the differential equation is

$$x_1' = f_1(x_1, x_2), \qquad x_2' = f_2(x_1, x_2),$$

the linearizatioin is $Z' = AZ$ with

$$A \;=\; \begin{pmatrix} (f_1)_{x_1} & (f_1)_{x_2} \\ (f_2)_{x_1} & (f_2)_{x_2} \end{pmatrix}\,.$$

For the present problem $f_1 = -V_{x_1}$ and $f_2 = -V_{x_2}$ so one has

$$A \;=\; -\begin{pmatrix} V_{x_1x_1}(\underline{x}_1, \underline{x}_2) & V_{x_1x_2}(\underline{x}_1, \underline{x}_2) \\ V_{x_2x_1}(\underline{x}_1, \underline{x}_2) & V_{x_2x_2}(\underline{x}_1, \underline{x}_2) \end{pmatrix}\,.$$

ii. Equality of mixed partials shows that A is a symmetric matrix. A symmetric matrix has only real eigenvalues.

iii. Computing the partial derivatives of V at the origin yields the matrix

$$A \;=\; \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}\,.$$

Since

$$\det(A - \lambda I) = \lambda^2 - 25,$$

the eigenvalues are ± 5.

The tangent to the stable manifold is the direction of the eigenvectors corresponding to the negative eigenvalue, equivalently the nonzero directions in

$$\ker(A + 5I) \;=\; \ker \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}\,.$$

These are the multiples of $(2, 1)$.

The tangent to the unstable manifold has the direction of

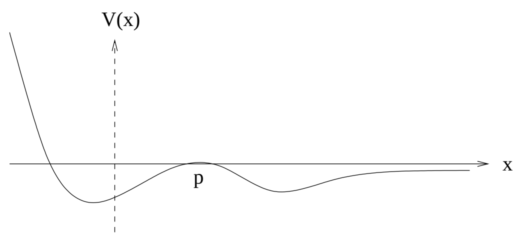
$$\ker(A-5I) \;=\; \ker\begin{pmatrix}-8 & 4\\ 4 & -2\end{pmatrix}$$

so the multiples of $(1,2)$. The directions are eigendirections of a symmetric matrix corresponding to distinct eigenvalues so must be orthogonal yielding a good check.

4. (2+3+3+1 points). Consider the system

$$x'' \;=\; -\frac{dV(x)}{dx}$$

where the graph of $V(x)$, sketched below, has three critical points and at each $V'' \neq 0$. The graph approaches the x axis from below as $x \to \infty$ and increases to $+\infty$ as $x \to -\infty$.



On orbits, the energy

$$E \;=\; \frac{(x'(t))^2}{2} \;+\; V(x(t))$$

is independent of time.

i. In the phase plane, sketch the level curves of energy that lie near each of the two local minima of V.

ii. Sketch the form of a level curve of energy corresponding to energy greater than the energy of the equilibrium solution at p. What is the shape of this level curve for x tending to ∞?

iii. Sketch the level curve of the energy whose energy is equal to the energy at the equilibrium $(p,0)$. What is the form near p and as $x \to \infty$? Explain.

iv. These level curves are orbits. Represent with an arrow the direction of motion on the orbits.

Solution. i. The energy has a strict local minimum at each of these two equilibria by the second derivative test for minima, the matrix of second derivatives of E at the minimum is equal to

$$\begin{pmatrix}E_{x_1x_1} & E_{x_1x_1'}\\ E_{x_1x_1'} & E_{x_1'x_1'}\end{pmatrix} \;=\; \begin{pmatrix}V'' & 0\\ 0 & 1\end{pmatrix}$$

The level curves near each strict minimum are therefore of roughly elliptical shape and are closed periodic orbits of the differential equation.

ii. The energy of the equilibrium at p is equal to zero. For $E > 0$ the level curve has equation

$$x' \;=\; \pm\sqrt{2(E - V(x))}\,.$$

It lies in the region $x \geq \underline{x}$ where $\underline{x} < 0$ is the unique value where $V(\underline{x}) = E$.

As $x \to \infty$, $V(x) \to 0$, so the curve in $x' > 0$ converges to $x' = \sqrt{2E}$ from above. In $x' < 0$ it converges to $-\sqrt{2E}$ from below.

iii. The level set with $E = 0$ passes through p. Taylor expansion of the energy about $x = p$ yields one the equation

$$\frac{(x')^2}{2} + V''(p)\frac{(x-p)^2}{2} + hot \; = \; 0\,.$$

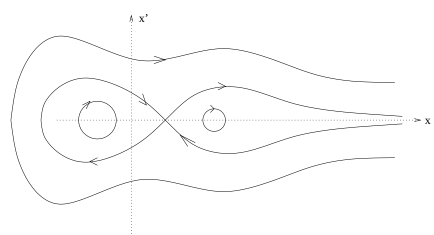
The higher order terms are cubic or higher. Dropping the higher order terms and using $V''(p) < 0$ yields

$$\Big(x' + \sqrt{|V''(p)|}\,(x-p)\Big)\Big(x' - \sqrt{|V''(p)|}\,(x-p)\Big) \;=\; 0.$$

This is the equation of the pair of lines $x' = \pm\sqrt{|V''(p)|}\,(x-p)$ in the x', x-plane. *Near p the energy zero level curve approximatly the cross formed by the intersecting lines* $x' = \pm\sqrt{|V''(p)|}\,(x-p)$.

Alternatively. These lines are the tangents to the stable and unstable manifolds . Linearization at the equilibrium shows that it is a saddle. Then computation of the eigenvectors to find the tangents to the stable and unstable manifolds. The analysis presented relies exclusively on a study of the energy function.

The energy zero level curve turns around the the left equilibrium and makes an "X" at the unstable equilibrium. It turns around the right hand equilibrium (where it is the limit of larger and larger periodic orbirs. As $x \to \infty$ the part in $x' > 0$ converges to the x axis from above while the part in $x' < 0$ converges to the x-axis from below.



iv. In the half plane $x' > 0$, x increases on orbits while in $x' < 0$ x decreases on orbits. This yields the directions of the arrows in the figure.

Discussion. The unstable manifold of p consists of a southwest branch that is a homoclinic orbit reconnecting to p as $t \to \infty$. It arrives at p as a branch of the stable manifold. The northwest branch of the unstable manifold goes to $x = +\infty$ where it converges to the x axis from above. The branch of the stable manifold that comes in from the southeast has come from $x = -\infty$.

Discussion. The equation $x' = \pm\sqrt{2E - V(x)}$ yields a perfectly smooth crossing of the x-axis on the left while it makes an "X" at the equilibrium p. This is a subtle point that can be viewed from several perspectives. It is a good idea to understand at least one.

5. (8 points). For the mechanical system

$$x'' + \big(1 + (x')^2\big)\, x' + x \;=\; 0 \tag{5.1}$$

with strong nonlinear friction show that all orbits converge to the equilibrium $x = 0, x' = 0$. **Hint.** The inspiration is the Lyapunov functional for linear friction together with La Salle's principles.

Solution. For the frictionless spring $x'' + x = 0$, the energy $((x')^2 + x^2)/2$ is constant on orbits. With linear friction, $x'' + ax' + x = 0$ with $a \geq 0$, energy decreases on orbits. Energy decreases even if $0 \leq a(t)$ depends on time. It is therefore likely that with the nonlinear friction, that energy will also decrease on orbits.

Motivated by this reasoning, introduce the candidate for a Lyapunov function

$$L(x, x') \;:=\; \frac{(x')^2}{2} \;+\; \frac{x^2}{2}\,.$$

Multiplying the differential equation by x' shows that on orbits

$$\frac{d}{dt} L(x(t), x'(t)) \;=\; -(1 + x'(t)^2)\, x'(t)^2 \;\leq\; 0\,. \tag{5.2}$$

L has a strict local minimum at the origin and is non increasing on orbits. It is NOT a strict Lyapunov function because $\dot{L}$ vanishes on the axis $\{x' = 0\}$ in the phase plane.

Lyapunov's theorems apply directly with this L to prove that the origin is stable but do not imply asymptotic stability.

It is not necessary to construct a strict Lyapunov function because La Salle's principles suffice with this Lyapunov function in spite of the fact that it is not strict.

For any point $\underline{x}, \underline{x}'$ of the phase plane, choose $R \geq L(\underline{x}, \underline{x}')$. The set $\mathcal{P} := \{L \leq R\}$ is the closed disk of radius $\sqrt{2}\,R$ in the x, x'-phase plane. It is closed and bounded. It is positive invariant because L decreases on orbits.

La Salle tells us that to show that all orbits in $\mathcal{P}$ tend to the origin it suffices to show that the origin is the only orbit in $\mathcal{P}$ along which L is constant.

Equation (5.2) shows that if L is constant on an orbit, then on the orbit x' must be identically equal to zero. Equivalently, x is constant. Since x and x' are constant the orbit is an equilibrium. The only equilibrium for (5.1) is $(x, x') = (0, 0)$. Thus the only orbit in $\mathcal{P}$ on which L is constant is the origin. This completes the demonstration.

System version. Instead of working with the second order equation thinking always of the x, x' phase plane, one can explicitly reduce to a first order system

$$x' = y, \qquad y' + (1 + y^2)y)x = 0.$$

For $L(x, y) := (x^2 + y^2/2$, compute $\dot{L} = -(1 + y^2)y^{\leq}0$. L has a strict global minimum at the origin so is a Lyapunov function. However, $\dot{L} = 0$ on the y-axis showing that L is NOT a strict Lyapunov function.

An orbit in $\mathcal{P}$ along which L is constant must lie on $y = 0$. Therefore $x' = y = 0$ so x is constant. Thus the orbit is $x = \text{constant}, y = 0$ so is an equilibrium in $\mathcal{P}$. The only equilibrium for this system is $(0, 0)$ completing the demonstration.

6. (4+4 points). Define the family of maps $f(a,x)$ from the real line to itself by

$$f(a,x) \;:=\; \frac{ax}{1+x^2}\,.$$

i. Find all fixed points for all values $-\infty < a < \infty$. Sketch them on a bifurcation diagram. **Ans.** The diagram is a pitchfork.

ii. Determine the stability of the fixed points and indicate the stability on the diagram with an s for stable or a u for unstable.

Solution. i. The fixed points are the solutions of

$$\frac{ax}{1+x^2} \;=\; x\,.$$

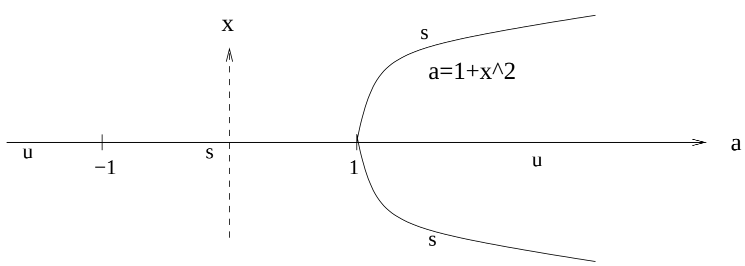
For all a, $x=0$ is a fixed point.

Fixed points with $x \neq 0$ are exactly the solutions of

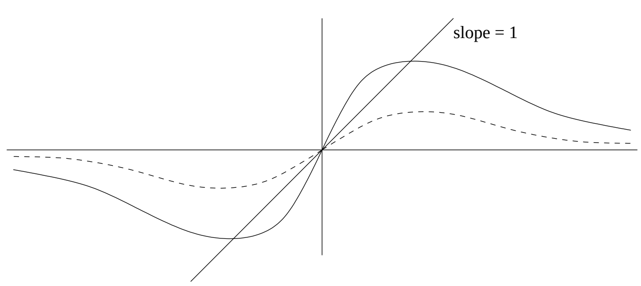
$$\frac{a}{1+x^2} \;=\; 1\,, \qquad \text{equivalently,} \qquad a = 1+x^2\,.$$

A second curve of fixed points is the parabola $\{a = 1+x^2\}$.

The two curves of equilibria are sketched below.



Supplement that was not requested. A graph of f for $1 > a > 0$ is indicated in dashes and for $a > 1$ solid. This the shows emergence of new fixed points.



ii. Next compute the stability. The answers are already indicated in the graph. *A fixed point is stable when $|f_x| < 1$ and unstable when $|f_x| > 1$.* Compute using the quotient rule,

$$f_x \;=\; \left(\frac{ax}{1+x^2}\right)' \;=\; \frac{(1+x^2)\cdot a - ax\cdot 2x}{(1+x^2)^2} \;=\; a\,\frac{1-x^2}{(1+x^2)^2}$$

On the curve $x = 0$ one has $f_x = a$ so the equilibrium is stable for $|a| < 1$ and unstable for $|a| > 1$

On the curve $a = 1 + x^2$, one has $f_x = (1 - x^2)/(1 + x^2)$ that is always less than one in magnitude except at the bifurcation point $x = 0$. Thus the extreme blades of the pitchfork are stable.

Discussion. There is a period doubling bifurcation at $a = -1$ that accounts for the loss of stability as a decreases through -1. The attracting fixed point present for $-1 < a < 1$ is replaced by an attracting two cycle for values of a just below -1. **Project.** Investigate further period doubling by searching graphically for 4-cycles.