Math 558, Fall 2012, Prof. J. Rauch. NAME ....

Midterm Exam Solutions, October 18, 2012

Instructions. 1. Two sides of a 3.5in. $\times$ 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are six questions.
5. You may use the back of the pages. Extra pages are available.

1. (6+1 points). **i.** Find the general solution of the differential equation

$$X' = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} X .$$

A series with an infinite number of non zero terms is **not** acceptable.

ii. Determine the stability of the equilibrium solution $(0,0)$.

Solution. i. Denote by A the coefficient matrix and compute the characteristic polynomial

$$A := \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \qquad \det(A - zI) = \det \begin{pmatrix} 3-z & -2 \\ 2 & -1-z \end{pmatrix} = (3-z)(-1-z)+4 = (z-1)^2 .$$

The only eigevalue is 1 and it has multiplicity equal to 2 as a root of the characteristic equation.

Find the eigenvectors by computing the kernel of

$$A - I = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} .$$

The kernel consists of vectors satisfying $x_1 = x_2$ that is the multiples of $(1,1)$. The kernel is one dimensional so up to scalar multiples there is a unique eigenvector, and the associated solution $e^t(1,1)$.

The generalized eigenspace

$$\ker(A-I)^2 = \ker 4 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}^2 = \ker \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2$$

is two dimensional.

Write e^{At} with $A = I + (A - I)$, the sum of two commuting matrices. Therefore,

$$e^{At} = e^{It}\, e^{(A-I)t} = e^t \Big[I + (A-I)t \Big] = e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \right],$$

where the other terms in the expansion of $e^{t(A-I)}$ vanish since $(A-I)^k = 0$ for $k \geq 2$.

The columns of e^{At} are independent solutions of the differential equation so the general solution is

$$c_1\, e^t \Big((1,0) + t(2,2) \Big) + c_2\, e^t \Big((0,1) + t(-2,-2) \Big)$$

with scalars c_1 and c_2.

Equivalently, the general solution is the product of e^{At} times (c_1, c_2) yielding the solution with value (c_1, c_2) at $t = 0$. This yields the same formula as above.

ii. Each solution of the differential equation diverges exponentially to infinity as $t \to \infty$ showing that even if you start arbitrarily close to the origin you diverge to infinity. The equilibrium 0 is unstable.

Remark. Some students have observed that for this 2×2 repeated eigenvalue case one can find the solution by calculating carefully $(At)^n$ then summing explicitly the infinite series that appear in the expansion of e^{At}. This simple idea will work for larger systems. There one could sum the series for initial data that belong to any of the generalized eigenspaces, but one must determine those first. And must know that they span. One needs the Spectral Theorem.

2. (5+ 2+3 points). **i.** Find the general solution of

$$X' \;=\; \begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix} X\,,$$

and show that it is of spiral type.

ii. Determine the direction of rotation about the equilibrium.

iii. Determine the principal axes of the associated elliptical orbits of $X' = \big(A - \big((\mathrm{tr}\,A)/2\big)\big)I\big)X$.

Solution. i. Denote again the constant matrix coefficient by

$$A = \begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix}, \qquad \det(A - zI) = \det\begin{pmatrix} 3-z & -1 \\ 5 & -1-z \end{pmatrix} = (3-z)(-1-z) + 5 = z^2 - 2z + 2\,.$$

The quadratic formula yields the eigenvalues,

$$z = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i\,.$$

This is a pair of complex conjugate eigenvalues with real part equal to one so orbits are of the form e^t times an elliptical orbit. They are exponentially growing spirals.

Compute the eigenvectors with eigenvalue $1+i$. They are the non zero vectors in the kernel of

$$A - (1+i)I = \begin{pmatrix} 3-(1+i) & -1 \\ 5 & -1-(1+i) \end{pmatrix} = \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}$$

The (x_1, x_2) belonging to the kernel are exactly those satisfying $(2-i)x_1 - x_2 = 0$, that is $(x_1, (2-i)x_1)$. Taking $x_1 = 1$ yields the solution

$$\Phi_1(t) \;=\; e^{(1+i)t}\,\big(1, 2-i\big)\,.$$

A solution associated to the complex conjugate eigenvalue is the complex conjugate of Φ_1 yielding

$$\Phi_2(t) \;=\; e^{(1-i)t}\,\big(1, 2+i\big)\,.$$

The general solution is

$$c_1\Phi_1 + c_2\Phi_2$$

with complex scalars c_j.

ii. The direction of the orbit through $(1,0)$ is $A(1,0) = (3,5)$ the first column of A. This points from $(1,0)$ into the positive quadrant showing that the spirals are traversed in the counterclockwise sense.

iii. Denote by B the traceless part of A

$$B \;:=\; A - \frac{\mathrm{tr}(A)}{2} I \;=\; \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}.$$

The equation $X' = BX$ has elliptical orbits and the question asks for the principal axes. Those axes are in the directions X that satisfy $0 = BX \cdot X$. Compute

$$\begin{aligned} BX \cdot X &= \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2, 5x_1 - 2x_2) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= (2x_1^2 - x_1x_2) + (5x_1x_2 - 2x_2^2) = 2(x_1^2 + 2x_1x_2 - x_2^2). \end{aligned}$$

When this vanishes with $X \neq 0$ one must have $x_2 \neq 0$. In that case dividing by x_2^2 and introducing $y := x_1/x_2$ yields $y^2 + 2y - 1 = 0$. The quadratic formula yields

$$y = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{2 \cdot 4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}.$$

Taking $x_2 = 1$ this yields the two direction

$$(-1 \pm \sqrt{2}\,,\, 1)$$

as the directions of the principal axes.

As a check verify that the directions are orthogonal,

$$(-1+\sqrt{2}, 1) \cdot (-1-\sqrt{2}, 1) = ((-1)^2 - \sqrt{2}^2) + 1 \;=\; 1 - 2 + 1 \;=\; 0.$$

3. (2+5 points). For small $0 < \epsilon << 1$ consider the initial value problem,

$$x' \;=\; x(1 + \epsilon \sin x), \qquad x(0,\epsilon) \;=\; 1$$

determining $x(t,\epsilon)$.

i. Find the unperturbed solution $x(t,0)$.

ii. Find an initial value problem determining the order ϵ corrector term, $z(t)$ in the perturbation theory approximation

$$x(t,\epsilon) \;\approx\; x(t,0) \;+\; \epsilon\, z(t) \;+\; \text{higher order terms}.$$

You need not solve the initial value problem for $z(t)$.

Solution i. The unperturbed solution comes from setting $\epsilon = 0$ in the initial value problem to find $x(t,0)' = x(t,0)$, $x(0,0) = 1$ with exact solution $x(t,0) = e^t$.

ii. Differentiate the differential equation with respect to ϵ. This is justified by the differentiable dependence on parameters part of the Fundamental Existence and Uniqueness Theorem.

$$\frac{\partial}{\partial\epsilon}\frac{\partial x}{\partial t} = \frac{\partial}{\partial\epsilon}\big(x(1+\epsilon\sin x)\big) = \frac{\partial x}{\partial\epsilon}(1+\epsilon\sin x)+x\,\frac{\partial}{\partial\epsilon}(1+\epsilon\sin x) = \frac{\partial x}{\partial\epsilon}(1+\epsilon\sin x)+x\sin x+x\,\epsilon\,\frac{\partial}{\partial\epsilon}\sin x\,.$$

In this use the equality of mixed partials, set $\epsilon = 0$ and $z(t) := \partial x/\partial\epsilon|_{\epsilon=0}$ to find $z' = z + x(t,0)\sin x(t,0) = z + e^t \sin e^t$. Differentiating the initial condition with respect to ϵ yields $z(0) = 0$ whence the Initial value problem

$$z' = z + e^t\sin e^t\,, \qquad z(0) = 0\,.$$

4. (5+1+1 points). Consider the scalar ordinary differential equation

$$x' \;=\; f(x)\,, \tag{1}$$

with $f(x)$ continuously differentiable. You are given the following information about the continuously differentiable function $f(x)$,

$$f(0) = f(10) = 0, \qquad \text{and} \quad f(7) < 0.$$

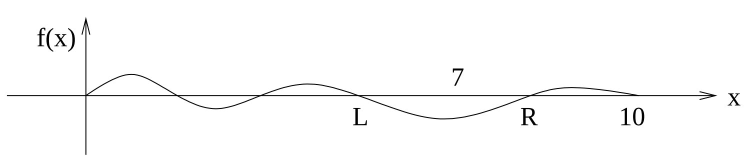
Warning. You are only told that f is negative at the single point 7. You might want to draw some figures.

i. Use the fundamental theorem of the phase line to show that the solution $x(t)$ of the initial value problem with $x(0) = 7$ is a decreasing function of time that exists for all $t > 0$ and satisfies $x(t) > 0$ for all $t > 0$. [1]

ii. The result in **i** implies that (you need not prove this) $\alpha := \lim_{t\to\infty} x(t)$ exists. Describe the solution that satisifes $x(0) = \alpha$.

iii. Can there be a point $\underline{x} \in]\alpha, 7[$ with $f(\underline{x}) = 0$? Why or why not?

Solution. i. The subtlety in this question is that you are given that f is negative at 7 and zero at 0 and 10 but you are not given that $f < 0$ on the entire interval $]0, 10[$. The function f can have other zeroes. There can be intervals where $f > 0$. On such intervals solutions of the differential equation would be increasing. You cannot directly apply the fundamental theorem of the phase line whose main hypothesis is recalled in the footnote. An example is sketch below.



The key idea is to consider the set E of equilibria in $[0, 10]$ that is the set of points $x \in [0, 10]$ so that $f(x) = 0$. It is closed set and $7 \notin E$. There are equilibria to the left of 7 including at least 0 and also to the right. There is a largest equlibrium to the left of 7. Call it L for left. Then $f(L) = 0$ and f has one sign on $]L, 7[$ so $f < 0$ on $]L, 7[$ since $f < 0$ near 7. Similarly if R is

[1] **Reminder.** The main hypotheses of the fundamental theorem are $f(a) = f(b) = 0$ and $f > 0$ on $]a, b[$.

the smallest equilibrium to the right of 7 then $f(R) = 0$ and $f < 0$ on $]7, R[$. Thus L and R are equilbria with $f < 0$ between. The fundamental theorem of the phase line then implies that the orbit with $x(0) = 7$ is decreasing exists for all time and satisfies

$$\lim_{t\to\infty} x(t) \ = \ L, \qquad \text{and} \qquad \lim_{t\to-\infty} x(t) \ = \ R\,.$$

This is more than asked for in **i.**

ii. The analysis above shows that $\alpha = L$ the first equilibrium to the left of 7. Therefore the solution with $x(0) = \alpha$ is the equilibrium that satisfies $x(t) = \alpha$ for all t.

iii. The answer NO is explained in **i.**

Alternatively, if there were such an $\underline{x}$ then one woulld necessarily have $x(t) > \underline{x}$ for all $t > 0$ since the solution can not touch the equilibrium in finite time. Then it would be impossible to have $x(t) \to \alpha$ as $t \to \infty$. This alternative explanation reproves part of the fundamental theorem of the phase line.

5. (3+2+3 points) Consider the scalar ordinary differential equation

$$x' \ = \ f(t, x)\,,$$

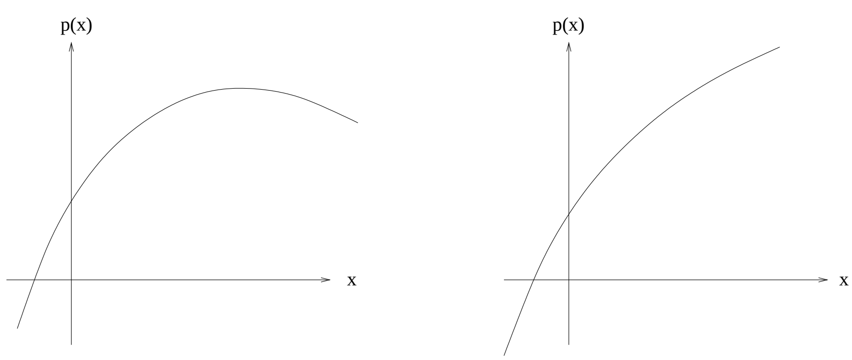
where f is continuously differentiable with respect to t, x and is periodic with period 1 in t. That is for all (t, x),

$$f(t + 1, x) \ = \ f(t, x)\,.$$

Suppose that it is known that solutions with arbitrary initial values exist for all time. Denote by p the Poincaré map that for a solutions $x(t)$ maps $x(0)$ to $x(1)$.

This question has parts **i**, **ii**, and **iii**.

i. Of the following two graphs, one **cannot** be the graph of a Poincaré map. Which one and why?

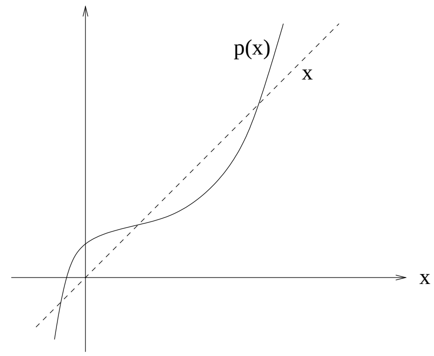


Solution i. The Poincaré map is known to be strictly monotone increasing. The function on the left is increasing to the left of the maximum and decreasing after so CANNOT be a Poincaré map.

Alternatively one could reprove a part of the monotonicity as follows. For the graph on the left if one considers a value y just a little bit lower than the maximum then there are points $x_1 < x_2$ so that $p(x_j) = y$ for $j = 1$ and $j = 2$. The points (x_1, y) and (x_2, y) are points on the graph of p to the left and right of the maximum at the same height y.

Then the orbits starting at $t = 0$ at the points x_1 and x_2 both reach the point y at time $t = 1$. This violates uniqueness of the solution with $x(1) = y$.

ii. If the graph below is the Poincaré map, how many periodic orbits with period equal to 1 are there? Explain.



Solution. The periodic orbits are exactly the solutions of $p(x) = x$, namely the solutions that return to their initial value after time 1. These are the points where the graph of p crosses the dashed line with slope 1 in the figure. There are exactly three points of intersection, therefore three periodic orbits of period 1.

iii. Pick and clearly identify one of the periodic orbits. Determine its stability. Explain.

Solution. Call the initial values of the periodic orbits $x_1 < x_2 < x_3$.

The graph shows that $p(x) > x$ on $]x_1, x_2[$ and $p(x) < x$ on $]x_2, x_3[$.

The Fundamental Theorem of Monotone Maps implies that the orbit of any point starting in $]x_1, x_2[$ is strictly increasing and converges to x_2 as $t \to \infty$. Similarly the orbits starting in $]x_2, x_3[$ are strictly decreasing and tend to x_2. This shows that the orbit starting at x_2 is stable.

6. (7 points). Consider

$$X' = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix} X := AX, \qquad \text{with,} \qquad \det(zI - A) = z\big(z - 2i\big)\big(z + 2i\big).$$

You are given the information that for the eigenvalue $2i$, $(0, i, 1)$ is an eigenvector. Find the general solution of the differential equation.

Solution. The given characteristic polynomial implies that the 3×3 matrix A has three distinct eigenvalues, 0, $2i$, and $-2i$. Therefore there is a basis of eigenvectors and to construct the general solution it is sufficient to find an eigenvector for each eigenvalue.

The given eignenvalue and eigenvector implies that

$$\Phi_1(t) := e^{2it}(0, i, 1)$$

is a solution.

Since the equation is real, the complex conjugate is also a solution, namely

$$\Phi_2(t) := e^{-2it}(0, -i, 1).$$

This solution is associated to the eigenvalue $-2i$ and eigenvector $(0, -i, 1)$.

Need to find eigenvectors associated to the eigenvalue 0. The eigenspace is the kernel of $A - 0I = A$. The kernel of A is defined by the pair of equations

$$3x_1 - 2x_3 = 0, \qquad \text{and} \qquad 2x_1 + 2x_2 = 0\,.$$

The kernel, parameterized by x_3 is given by,

$$x_1 = (2/3)x_3, \qquad \text{and,} \qquad x_2 = -x_1 = -(2/3)x_3\,.$$

Taking $x_3 = 3/2$ yields $(1, -1, 3/2)$. Multiplying by 2 yields the simpler $(2, -2, 3)$ and corresponding solution

$$\Phi_3(t) = e^{0t}(2, -2, 3) = (2, -2, 3)\,.$$

The general solution is

$$c_1\Phi_1(t) + c_2\Phi_2(t) + c_3\Phi_3(t)$$

where the c_j are complex scalars.