Math 558. Fall 2013. Prof. J. Rauch. NAME ....

Final Exam Solutions December 18, 2013

Instructions. 1. Two sides of two 3.5in. × 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, . . . etc.
3. Show work and explain clearly.
4. If you apply theorems from the course state them clearly and verify the hypotheses.
5. There are 6 questions, 6 pages, and a total of 47 points.
6. You may use the back of the pages and/or supplementary sheets.

1. (4+1+4 points). The origin $X = 0$ is an equilibrium of the the system

$$x' \;=\; x - 2y + 3z^2\,, \qquad y' \;=\; y + 4z - x^2\,, \qquad z' \;=\; -z + y^2\,.$$

i. Determine whether the origin is stable or not.

ii. Determine the dimension of the stable manifold.

iii. Determine the tangent plane at the origin to the stable manifold.

Solution. i. The linearization at the origin comes from dropping the quadratic terms yielding the 3×3 linear system with matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

The matrix is upper triangular so the eigenvalues are the diagonal elements, 1,1,-1. As there are eigenvalues in the right half plane the equilibrium is **unstable**.

ii. The dimension of the stable manifold is equal to the total multiplicity of eignevalues with strictly negative real part. In this case this equals **one**.

iii. The tangent space is spanned by any nonzero eigenfunction with eigenvalue -1.

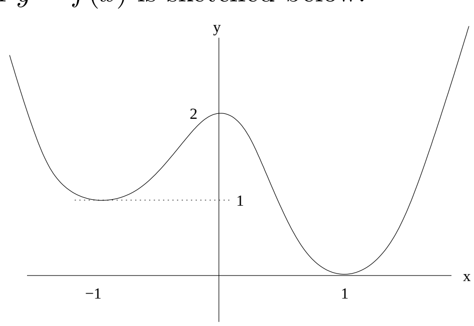
$$A - (-1)I \;=\; \begin{pmatrix} 2 & -2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvecspace consists of vectors satisfying

$$x - y = 0, \qquad 2y + 4z = 0\,.$$

This is a one dimensional space spanned, for example, by the solution that has $z = 1$, that is $x = y = -2$. The tangent space to the stable manifold has tangent direction $(-2, -2, 1)$.

2. (4+4 points). The graph of $y = f(x)$ is sketched below.



Consider the family of differential equations parameterized by a,

$$x' \;=\; f(x) - a\,, \qquad -\infty < a < \infty\,.$$

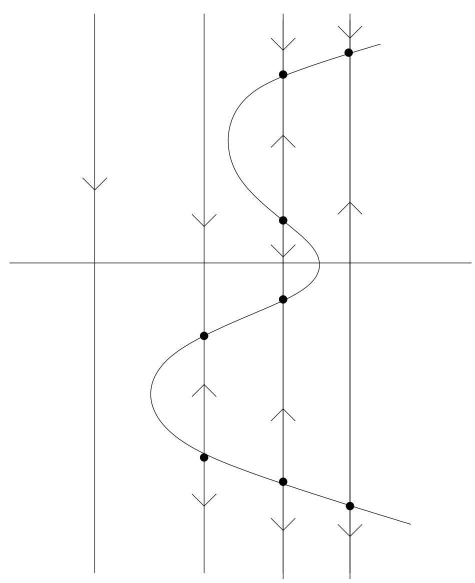
i. (Bifurcation diagram.) In the $(a, x)-$plane with the a-axis horizontal, sketch the set of equilibria as functions of a.

ii. As a varies, there are qualitatively different phase portraits that occur separated by bifurcation values of a. Sketch the phase portraits that occur between the bifurcation values. Indicate the regions of a where they occur.

Solution. i. For a fixed the equilibria are the points x so that $f(x) = a$. These are the intersection points of the graph $y = f(x)$ and the horizontal line $y = a$. Thus

- For $a < 0$ there are no equilibria.
- For $0 < a < 1$ there are two equilibria.
- For $1 < a < 2$ there are four equilibria.
- Foe $2 < a < \infty$ there are two equilibria.
- The bifurcation values are $a = 0, 1, 2$. At these values the lines $y = a$ are tangent to $y = f(x)$.

In the next diagram *the direction of increasing x is downward.*



The direction of motion on the phase line is computed from the sign of $f(x) - a$ and is indicated on the bifurcation diagram by an arrow.

3. (7 points). Consider the gradient system

$$x_1' = -V_{x_1}(x_1, x_2), \qquad x_2' = -V_{x_2}(x_1, x_2)$$

with potential $V := (x_1 + x_2)^2 + (x_1 - x_2)^4$. Show that every solution $X(t)$ approaches the origin as $t \to \infty$. Attn. Instruction 4.

Solution. This is an application of Lasalle's Theorem on basins of attraction.

The equation reads

$$x_1' = -2(x_1+x_2) - 4(x_1-x_2)^3\,, \qquad x_2' = -2(x_1+x_2) + 4(x_1-x_2)^3\,.$$

Equilibria must satisfy

$$-2(x_1+x_2) - 4(x_1-x_2)^3 = 0\,, \qquad --2(x_1+x_2) + 4(x_1-x_2)^3 = 0$$

whence $x_1 + x_2 = x_1 - x_2 = 0$. The only equilibrium is the origin.

Introduce the Lyapunov function $L(X) := V(X)$ that has a strict local minimum at the origin. The gradient of V vanishes only at the origin. Therefore L is strictly decreasing on orbits other than the equilibrium.

Suppose that $X(t)$ is a solution. Define $a := V(X(0))$. Then

$$\mathcal{P} \ := \ \Big\{ X \ : \ V(X) \le a \Big\}$$

is closed bounded and postively invariant. And, $X(0) \in \mathcal{P}$.

The only orbit in $\mathcal{P}$ on which L is constant is the equilibrium since at all other points $dL/dt = -\|\mathrm{grad}\, V\|^2 < 0$ is strictly decreasing.

Lasalle's Theorem implies that all oribits starting in $\mathcal{P}$ converge to the origin. Since $X(0) \in \mathcal{P}$ this proves that $\lim_{t\to\infty} X(t) = 0$. ∎

Discussion. The linearization is

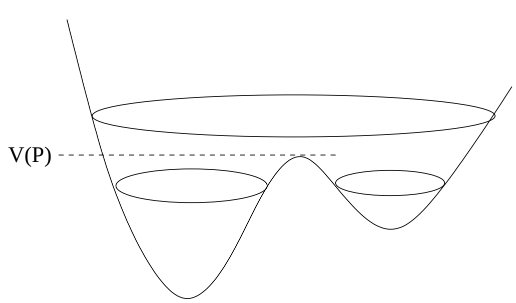
$$Y' \ = \ \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix} Y\,.$$

This implies nothing about stability. Asymptotic stability follows once the strict Lyapunov function is observed. The Lasalle Theorem shows that 0 is a global attractor. The same proof works for any V that has a single critical point that is a global minimum and $V \to \infty$ as $|X| \to \infty$.

4. (4+3 points). The gradient system

$$X' \ = \ -\,\mathrm{grad}\, V(X)\,.$$

has $V(x_1, x_2)$ with the following graph.



There are two local minima separated by a mountain pass through the point $(P, V(P))$ that is a critical point (def: a point where the gradient vanishes) of V but neither local maximum nor local minimum. The matrix of second derivatives of V at each critical point is invertible.

i. Describe the phase portrait of the linearizaton at the equilibrium P.

ii. How do you know that there is a non equilibrium orbit $X(t)$ with $\lim_{t\to\infty} X(t) = P$?

Solution. i. The linearization is the equation

$$Y' \;=\; AY$$

where A is the matrix of second derivative $A := \partial^2 V(P) \partial x_i \partial x_j$

A is a symmetric 2×2 matrix so has only real eigenvalues. By hypothesis A is invertible so the eigenvalues are non zero.

Since V has neigher a local minimum nor a local maximum there must be eigenvalues of both signs. Therefore there is one positive and one negative eigenvalue.

Since there is one positive and one negative eigenvalue the phase portrait of the linearization is a saddle.

ii. The Stable Manifold Theorem in this situation asserts that the set of initial data $\underline{X}$ with the property that the orbit through $\underline{X}$ approaches $(0, 0)$ as $t \to \infty$ is a smooth curve with tangent at the origin parallel to the eigendirection of A corresponding to the negative eigenvalue. There are exactly two solution curves approaching the origin. They come in from opposite directions.

5. (1+ 2+4+3 points). Consider the one dimensional mechanical system

$$x'' \;=\; -\frac{dV(x)}{dx}\,, \qquad V(x) \;:=\; \frac{-\,x^2}{1+x^2}$$

On orbits, the energy

$$E = \frac{(x')^2}{2} \;+\; V(x) \tag{$\dagger$}$$

is independent of time.

i. Find all equilibria.

ii. Show that for solutions with positive energy, of $|x'| \geq \sqrt{2E}$ and the orbits move steadily toward infinity.

iii. Sketch a typical integral curve with negative energy.

iv. Find the equations of the stable curves that converge to the equilibrium as $t \to \infty$.

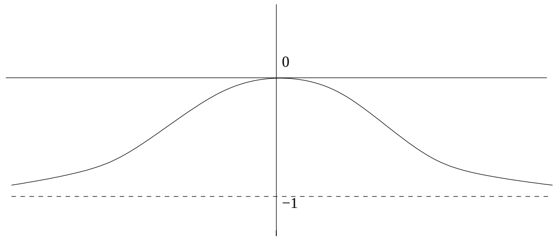
Solution. i. Near the origin $V(x) \approx x^2$ has a local maximum. Compute

$$V'(x) \;=\; \frac{(1+x^2)(-2x) - (-x^2)(2x)}{(1+x^2)^2} \;=\; \frac{-2x}{(1+x^2)^2}\,.$$

- As $x \to \infty$, $V(x)$ decreases to -1.
- $V'(x) > 0$ on $]-\infty, 0[$.

• $V'(x) < 0$ on $]0, \infty[$.

The graph resembles



The only equilibrium is $x = 0$.

ii. The orbits in the x, v phase space are given by the equation (†) namely

$$v^2 \;=\; 2(E - V(x))\,.$$

Therefore

$$|x'|^2 \;=\; 2(E - V(x) \;\geq 2E$$

since $-V \geq 0$. Proving the desired inequality.

Since x' is continuous and bounded away from zero in absolute value it cannot have both positive and negative values so either

$$x' \geq \sqrt{2E} \qquad \text{or} \qquad x' \leq -\sqrt{2E}\,.$$

In the first case $x \to \infty$, $V(x) \to -1$ and $v = x' \to \sqrt{2(E+1)}$. The curve in the phase plane has horizontal aysmptote $v = \sqrt{2(E+1)}$. The second case has horizontal asymptote $v = -\sqrt{2(E+1)}$.

iii. Since $V \geq -1$, the energy E of all orbits is at least -1.

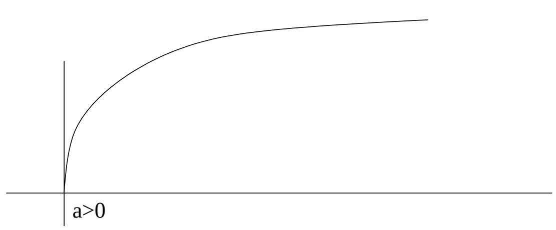
For $-1 < E < 0$ the symmetric interval $]-a, a[$ containing the origin on which $V(x) > E$ are points that cannot be reached by orbits since (†) cannot be satisfied. Therefore the orbits must satisfy either $x \geq a$ or $x \leq -a$. In either case the velocity satisfies

$$v \;=\; \pm\sqrt{2(E - V(x))\,.} \tag{††}$$

Since the curve $y = E$ and the graph $y = V(x)$ cross transversally at $x = \pm a$, the derivative of $E - V(x)$ at this point is nonzero so for x larger than but close to a one has

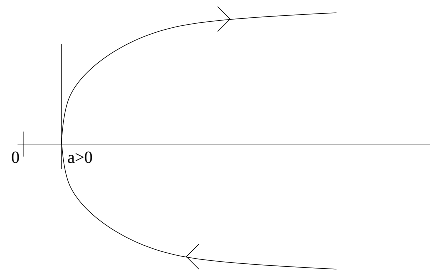
$$2(E - V(x)) \approx \alpha(x - a)\,, \quad \alpha > 0\,.$$

The square root therefore has vertical tangent and the curve (††) with the plus sign resembles



As $x \to +\infty$ the curve has a horizontal asymptote at height $v = \sqrt{2(E+1)}$.

The curve together with its mirror reflection in the horizontal axis gives a typical integral curve with energy between -1 and 0.



A second typical case is the reflection of these orbits in the v-axis.

iv. The limiting orbits with $E = 0$ have equations

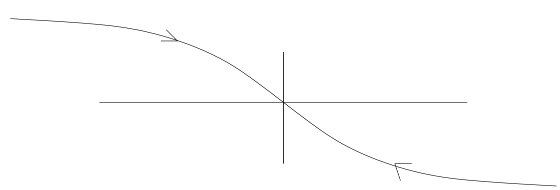
$$x' \;=\; \pm\sqrt{2(-V(x)}\,.$$

At the origin they are tangent to $v = \pm(\sqrt{2})x$ since

$$-V(x) \;=\; x^2 + \text{h.o.t}$$

The stable manifold is the union

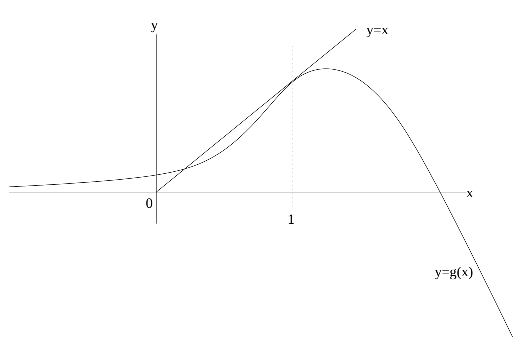
$$\Big\{(x,v)\,:\,x>0,\quad v=-\sqrt{-2V(x)}\Big\}\ \cup\ \Big\{(x,v)\,:\,x\le 0,\quad v=\sqrt{-2V(x)}\Big\}\,.$$



The motion is toward the origin on both branches since in $v > 0$, x increases and in $v < 0$, x decreases.

The left hand branch has asymptote $v = \sqrt{2}$. The right hand branch has asymptote $v = -\sqrt{2}$.

6. (5+4 points). The graph $y = g(x)$ is as follows.



Study the interates of the family of maps $h(a,x)$ from the real line to itself that are parameterized by a

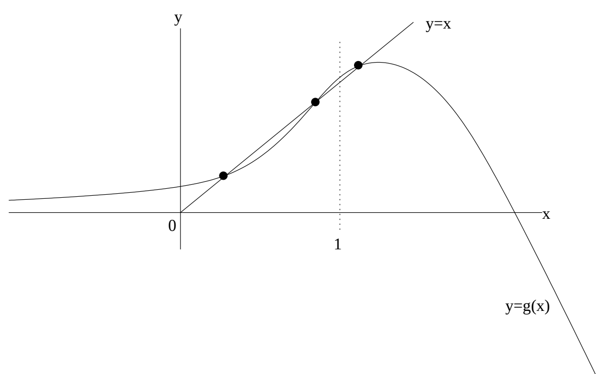
$$h(a,x) \;:=\; g(x) + a\,, \qquad -\infty < a < \infty\,.$$

i. Explain why $a = 0$ is a bifurcation point for fixed points and describe the bifurcation that occurs near $a = 0$. Include the stabilities of the fixed points in the discussion.

ii. Use the graph to find the second bifucation value a_c and describe the bifurcation that occurs near $a = a_c$. Include the stabilities of the fixed points in the discussion.

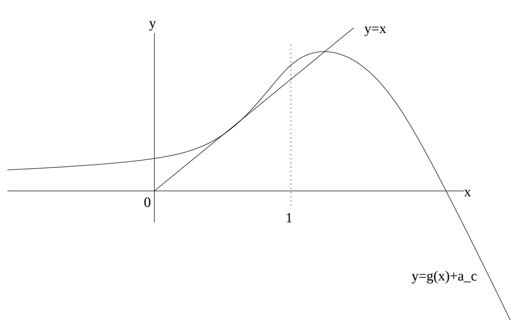
Solution. i. For $a < 0$ the curve $y = h(a,x)$ crosses the diagonal $y = x$ at a unique point $x(a)$. The slope of the curve at crossing is positive and less than one so the equilibrium is stable.

At $a = 0$ in addition to the stable orbit from the crossing at an $x \in]0,1[$, the curve touches the diagonal at $(1,1)$ so $x = 1$ is a second fixed point. For a just a little bigger than 0, there will be two fixed points near one. One to the left of 1 and one to the right.



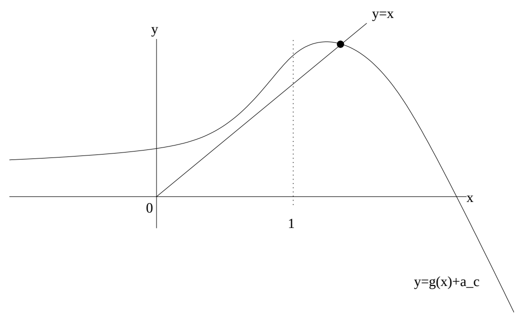
At the left of the newly bifurcated pair, the the slope of h slightly greater than one so it is unstable. At the right one the slope of h is slightly less than one so the fixed point is stable.

ii. As a continues to increase, the graph of h moves up and the stable fixed point on the extreme left and the unstable fixed point that is the middle one of the three approach each other till at the critical value a_c they coalesce and the graph of $g(x) + a_c$ is tangent to the diagonal from above as in the figure.



In a sense two fixed points annihilate.

For larger values of a only the stable largest fixed point persists.



Discussion. **1.** As a continues to increase the slope of h at the crossing goes negative and eventually will pass through -1 at which value of a there will be a period doubling. This is a *third* bifurcation. The fixed point will turn unstable and the newly born 2-cycle will be stable.

2. Denote by $0 < M < 1$ the point where $g'(M) = 1$. At that value of x the graph of $g(x)$ is below the diagonal by an amount $M - g(M) > 0$. The critial value a_c is given by the formula $a_c = M - g(M)$. The graph is drawn so that a_c is close to zero so this third bifurcation occurs after a_c.