Math 558. Fall 2013. Prof. J. Rauch. NAME ....

Midterm Exam, October 17, 2013

Instructions. 1. Two sides of a 3in. × 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ...etc.
3. Show work and explain clearly.
4. There are six questions.
5. You may use the back of the pages. Extra pages are available.

1. (8 points). Consider the system

$$X' = AX, \qquad A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 5 \end{pmatrix}.$$

You are given that $\det(A - zI) = (z-1)^2(5-z)$ and that for the eigenvalue 5, $(0,0,1)$ is an eigenvector. Find three linearly independent solutions and therefore the general solution. Solutions in terms of infinite series are not admissable.

Solution. The given data provides one solution

$$\Phi_1(t) := e^{5t}\,(0,0,1).$$

Need to find two independent solutions associated to the double root $\lambda = 1$. Compute

$$A - I := \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{pmatrix}.$$

There are two linearly independent rows so the kernel has dimension equal to one. The eigenspace has dimension equal to one. Need to use generalized eigenvectors.

Compute

$$(A-I)^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -12 & 6 & 16 \end{pmatrix}.$$

The kernel is two dimensional consisting of vectors satisfying $-6x_1 + 3x_2 + 8x_4 = 0$. An example of a basis for this two dimensional space is the independent pair of vectors $(0,8,-3)$ and $(4,0,3)$. Denote by X this two dimensional space of generalized eigenvectors. For $v \in X$, one has

$$e^{tA}v = e^t\Big(I + (A-I)t\Big)v = e^t\left(I + \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{pmatrix} t\right)v.$$

Three linear independent solutions are the given one together with the two new ones

$$\Phi_2(t) := e^t\Big(I + (A-I)t\Big)(0,8,-3) = e^t\left(I + \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 4 \end{pmatrix} t\right)(0,8,-3),$$

$$\Phi_3(t) \;:=\; e^t\Big(I+(A-I)t\Big)(4,0,3) \;=\; e^t\left(I+\begin{pmatrix}0&2&0\\0&0&0\\-3&3&4\end{pmatrix}t\right)(4,0,3)\,.$$

2. (4+ 2+4 points). **i.** Show that the phase portrait of

$$X' \;=\; A\,X\,,\qquad A\;:=\;\begin{pmatrix}3&5\\-2&-2\end{pmatrix}$$

is of spiral type.

ii. Determine the direction of rotation about the equilibrium and the stability of the equilibrium.

iii. Determine the principal axes of the associated elliptical orbits of

$$X' \;=\; \Big(A-\frac{\mathrm{tr}\,A}{2}\,I\Big)X\,.$$

Solution. i. Compute

$$\det\big(A-\lambda I\big) \;=\; \det\begin{pmatrix}3-\lambda&5\\-2&-2-\lambda\end{pmatrix} \;=\; (3-\lambda)(-2-\lambda)+10 \;=\; \lambda^2-\lambda+4\,.$$

The quadratic formula yields the two eigenvalues

$$\lambda_\pm \;:=\; \frac{1\pm\sqrt{1-16}}{2} \;=\; \frac{1\pm i\sqrt{15}}{2}\,.$$

The pair of complex conjugate eigenvalues with postive real part yields a phase diagram that spirals out.

ii. The postive real part shows that solutions grow exponentially as $t\to\infty$ and converge to 0 as $t\to-\infty$. *The equilibrium is unstable.*

The direction of the orbit at the point $(1,0)$ is equal to

$$\begin{pmatrix}3&5\\-2&-2\end{pmatrix}(1,0) \;=\; (3,-2)\,.$$

The second component is negative, the orbit points downward. *The direction of rotation is clockwise.*

iii. The traceless part is

$$A-\Big(\frac{\mathrm{tr}(A)}{2}\Big)I \;=\; A-I/2 \;=\; \begin{pmatrix}2.5&5\\-2&-2.5\end{pmatrix}\,.$$

The vectors X in the direction of the ellipse axes satisfy

$$\begin{aligned}0=2AX\cdot X \;&=\; \begin{pmatrix}5&10\\-4&-5\end{pmatrix}X\;\cdot\;X \;=\; \big(5x_1+10x_2\,,\,-4x_1-5x_2\big)\;\cdot\;(x_1,x_2)\\ &=5x_1^2+10x_1x_2-4x_1x_2-5x_2^2 \;=\; 5x_1^2+6x_1x_2-5x_2^2\,.\end{aligned}$$

There are no nontrivial solutions with $x_1 = 0$ so can divide by x_1^2 to show that this is equivalent to $y = x_2/x_1$ satisfying $5 + 6y - 5y^2 = 0$. The quadratic formula yields the two solutions

$$y_\pm \;:=\; \frac{-6 \pm \sqrt{36 + 4(25)}}{10}\,.$$

The axes point in the directions $(1, y_+)$ *and* $(1, y_-)$.

3. (2+5 points). For small $0 < \epsilon << 1$ consider the initial value problem,

$$x' \;=\; x + \epsilon\, x^4 \qquad x(0,\epsilon) \;=\; 1$$

determining $x(t,\epsilon)$.

i. Find the unperturbed solution $x(t,0)$.

ii. Find an initial value problem determining the order ϵ corrector term, $z(t)$ in the perturbation theory approximation

$$x(t,\epsilon) \;\approx\; x(t,0) \;+\; \epsilon\, z(t) \;+\; \text{higher order terms}\,.$$

You need not solve the initial value problem for $z(t)$.

Solution. i. The solution for $\epsilon = 0$ satisfies the initial value problem

$$x' = x\,, \qquad x(0) = 1$$

with solution $x(t,0) = e^t$.

ii. Derive an initial value problem satisfied by $\partial x(t,\epsilon)/\partial\epsilon$ by differentiating the equations defining $x(t,\epsilon)$ with respect to ϵ to find

$$\Big(\frac{\partial x}{\partial \epsilon}\Big)' \;=\; \frac{\partial x}{\partial \epsilon} \;+\; x^4 \;+\; \epsilon\frac{\partial (x^4)}{\partial \epsilon}\,, \qquad \frac{\partial x(0,\epsilon)}{\partial \epsilon} \;=\; 0\,,$$

valid for all t and ϵ for which the solution exists. Set $\epsilon = 0$ to derive the initial value problem for

$$z(t) \;:=\; \frac{\partial x}{\partial \epsilon}\Big|_{\epsilon=0}\,,$$

$$z' \;=\; z \;+\; e^{4t}\,, \qquad z(0) \;=\; 0\,.$$

4. (3+3 points). Consider two solutions of the the scalar ordinary differential equation

$$x' \;=\; f(t,x)\,, \qquad \widetilde{x}' \;=\; f(t,\widetilde{x}) \tag{1}$$

with $f(t,x)$ continuously differentiable and

$$x(0) \;<\; \widetilde{x}(0)\,.$$

You are given that the solutions exist for all $t \ge 0$. Explain why $x(t) < \widetilde{x}(t)$ for all $t > 0$.

Solution. If the conclusion were not true, there would be a $T > 0$ with $x(T) \geq \widetilde{x}(T)$.

Then $x(t) - \widetilde{x}(t)$ would be a continuous function on $0 \leq t \leq T$ that is strictly negative at $t = 0$ and nonnegative at $t = T$. Therefore there would be a time $0 < \underline{t} \leq T$ at which $x(\underline{t}) - \widetilde{x}(\underline{t}) = 0$.

The two function $x(t)$ and $\widetilde{x}(t)$ satisfy the same differential equation and have the same value at time $\underline{t}$. The fundamental uniqueness theorem implies that the two functions must be equal throughout their common domain of existence. In particular they would be equal at $t = 0$. This contradicts the hypothesis that $x(0) < \widetilde{x}(0)$.

Remark. The uniqueness theorem is used to pass from equality at time $T > 0$ to equality at time 0. This is uniqueness in the past.

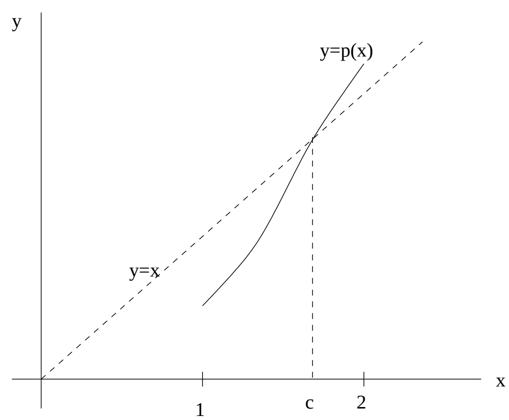
5. (2+3+3 points) Consider the scalar ordinary differential equation

$$x' \;=\; f(t, x)\,,$$

where f is continuously differentiable with respect to t, x and is periodic with period 1 in t. That is for all (t, x),

$$f(t+1, x) \;=\; f(t, x)\,.$$

Suppose that it is known that solutions with arbitrary initial values exist for all time. Denote by p the Poincaré map that for a solutions $x(t)$ maps $x(0)$ to $x(1)$. The graph of $y = p(x)$ for $1 \leq x \leq 2$ as well as the graph of $y = x$ are sketched below.



i. What can you say about the solution with initial value c?

ii. If a solution has initial data slightly larger than c, does the solution stay close to c as $t \to \infty$? Explain.

iii. If one slightly perturbs the periodic function f leaving it periodic in t with period 1, what is the expected qualitative behavior of solutions that start in $1 < x < 2$?

Solution. i. The graph shows that $p(c) = c$. The solution with initial value c returns after one period to its starting value c. This implies that the solution with inital value equal to c is a periodic function of period equal to 1.

ii. For initial values in the interval $]c, 2[$ the graph shows that $p(x) > x$. Thus after each period, the value of the solution increases. They will increase till they leave the region sketched. The Fundamental Theorem of Monotone Maps shows that if the graph of $p(x)$ crosses $y = x$ at a value above 2 then the orbit will converge to the periodic solution above and closest to c. Otherwise the solution diverges to infinity as $t \to \infty$. In either case the periodic orbit through c is unstable.

Remark. It is not enough to observe that the orbit $p^{(n)}$ is increasing to conclude instability.

iii. The graph of $p(x)$ crosses $y = x$ transversally. If one slightly perturbs the function f then the solutions of the differential equation are only slightly perturbed and therefore the graph of the poincaré map p is slightly perturbed. Therefore there will be a unique solution of $p(x) = x$ in the interval $]1, 2[$ and that solution will be near c and the graph will cross transverally from below to above. In particular the corresponding periodic solution remains unstable. This is an example without bifurcation. It is structurally stable.

Discussion. Small perturbations of f measure in the C^1 topology yield small perturbation of p in this topology. Then, an appeal to the implicit function theorem applied to the equation $p(x) = x$, yields a rigorous version. Such niceties of proof are not required of 558 students.

6. (3+2+3 points). The equation

$$x' \;=\; (x-1)(x-2)$$

has equilibria at $x = 1$ and $x = 2$.

i. Find the linearized equation at the equilibrium 1.

ii. Is the solution 0 a stable or unstable solution *of the linearized equation*?

iii. Explain how this determines the stability of the equilibrium 1.

Solution. At a solution $x(t)$ the linearized equation is the equation

$$y' \;=\; \frac{\partial f(x(t))}{\partial x}\, y\,.$$

Compute

$$\frac{\partial f(x)}{\partial x} \;=\; (x-1)\frac{\partial}{\partial x}(x-2) + \frac{\partial(x-1)}{\partial x}(x-2)$$

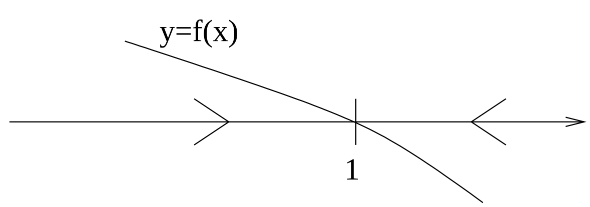
In case of the equilibrium $x(t) = 1$ the first term is eliminted to yield $f_x(1) = -1$ and the linearization

$$y' = -y\,.$$

ii. The solutions of the lienarized equation are $y = y(0)e^{-t}$ showing that $y = 0$ is asymptotically stable as a solution of the linearization.

iii. This shows that small perturbations will tend to decay, indicating stability.

Alternatively, $f(1) = 0$ and $f'(1) < 0$ imply that the graph of f crosses the x-axis transversally from postive to negative at $x = 1$ with corresponding part of the phase line sketched below.



The fundamental theorem of the phase line then implies asymptotic stability.