Math 558. Fall 2014. Prof. J. Rauch. NAME ....

Final Exam Solutions December 16, 2014

Instructions. 1. Two sides of two 3.5in. $\times$ 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ... etc.
3. Show work and explain clearly.
4. If you apply theorems from the course state them clearly and verify the hypotheses.
5. There are 5 questions, 5 pages, and a total of 61 points.
6. You may use the back of the pages and/or supplementary sheets.

1. (3+4+3=10 points). The origin $\underline{X} = 0$ is an equilibrium of the system

$$x' \;=\; x + 2z + yz\,, \qquad y' \;=\; y - xz\,, \qquad z' \;=\; 2x + z + xy\,.$$

i. Compute the linearized equation at $\underline{X} = 0$.

ii. Verify that the equilibrium is hyperbolic and determine the dimensions of the stable and unstable manifolds.

iii. Determine the tangent space at $\underline{X}$ to the stable manifold.

Solution. i. Since the equilibrium is $\underline{X} = 0$ the linearization comes from dropping quadratic and higher order terms in x, y, z so is

$$Y' \;=\; \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} Y \;:=\; AY\,.$$

ii. Compute the eigenvalues of the matrix A. Since A is symmetric, the eigenvalues are real. Expand the determinant by the first row to find

$$\det(A - zI) = \det \begin{pmatrix} 1-z & 0 & 2 \\ 0 & 1-z & 0 \\ 2 & 0 & 1-z \end{pmatrix} = (1-z)^3 + 2(-1)(1-z)(2) = (1-z)[(1-z)^2 - 4].$$

Set this equal to zero. The roots are the eigenvalues. The first factor shows that $z = 1$ is an eigenvalue. The second factor yields

$$(1-z)^2 = 4, \qquad 1 - z = \pm 2, \qquad z = 3, -1\,.$$

The eigenvalues are 3,1,-1.

Since none lie on the imaginary axis the equilibrium is hyperbolic.

The dimension of the unstable manifold is equal to the number of eigenvalues with positive real part, therefore equal to 2.

The dimension of the stable manifold is equal to the number of eigenvalues with negative real part, therefore equal to 1.

iii. The tangent to the stable manifold is the span of the generalized eigenvalues corresponding to eigenvalues with negative real part.

The only such eigenvalue is -1 and it has multiplicity equal to 1. The direction of the tangent line is that of eigenvectors with eigenvalue -1. The eigenvectors are the nonzero vectors in the kernel of

$$A-(-1)I=\begin{pmatrix}2&0&2\\0&2&0\\2&0&2\end{pmatrix}$$

This matrix has rank two so the kernel has dimension equal to 1. The kernel is the set of (x,y,z) so that

$$x+z=0\qquad\text{and}\qquad y=0\,.$$

These are the vectors $\mathbb{R}(1,0,-1)$. Any nonzero multiple of $(1,0,-1)$ is tangent to the one dimensional unstable manifold (curve) at $(0,0,0)$.

2. (4+2+2+2+3=13 points). Consider the following two systems;

$$\begin{aligned}x_1' &= -(x_1-1)\ +\ (x_2-2)\ -\ (x_1-1)(x_2-2)\\ x_2' &= 2\,(x_2-2)\ +\ (x_1-1)^2\end{aligned}\tag{$\mathbf{a}$}$$

$$\begin{aligned}x_1' &= -2\,(x_1-1)\ +\ (x_1-1)(x_2-2)\\ x_2' &= 2\,(x_2-2)\ +\ (x_1-1)^2\end{aligned}\tag{$\mathbf{b}$}$$

Both have the equilbrium point $\underline{X}=(1,2)$. Neither has $(2,2)$ as an equilibrium point. Instruction 4 is particularly pertinent for ii, iii, and iv.

i. Compute the linearized equations at $\underline{X}=(1,2)$.

ii. Are systems **a** and **b** differentiably conjugate on a neighborhood of $(1,2)$?

iii. Are systems **a** and **b** topologically conjugate on a neighborhood of $(1,2)$?

iv. Are systems **a** and **b** differentiably conjugate on a neighborhood of $(2,2)$?

v. Suppose that $X'=F(a,X)$ is a continuously differentiable family of equations depending on the real parameter a and reducing to system (**a**) when $a=0$. Explain why $a=0$ is NOT a bifurcation point. That is, the only equilibria near $(1,2)$ for $a\approx 0$ lie on a curve $X=K(a)$ with K a continuously differentiable function of a satisfying $K(0)=(1,2)$.

Solution. i. The linearization can be read off from the equations as they are written. One need only neglect the terms higher order in $x-1$ and $x-2$. This yields the linearizations

$$y_1'\ =\ -\,y_1\ +\ y_2\,,\qquad y_2'\ =\ 2\,y_2\tag{$\mathbf{a},\mathbf{linearized}$}$$

$$y_1'\ =\ -2\,y_1\,,\qquad y_2'\ =\ 2\,y_2\tag{$\mathbf{b},\mathbf{linearized}$}$$

ii. The matrices for the linearized systems are

$$\mathbf{a}.\quad\begin{pmatrix}-1&1\\0&2\end{pmatrix},\qquad\mathbf{b}.\quad\begin{pmatrix}-2&0\\0&2\end{pmatrix}.$$

The eigenvalues for (**a**) are -1 and 2, the eigenvalues for (**b**) are 2 and -2. Since the eigenvalues are not the same the systems are not linearly conjugate. **Discussion.** Linear conjugacy of linearizations is a necessary condition for differentiable conjugacy. It is not a sufficient condition. Some students missed this.

If system (**a**) and (**b**) were differentiably conjugate on a neighborhood of the equilibrium $(1, 2)$, then their linearizations would be linearly conjugate. Since the linearizations are not linearly conjugate we conclude that systems (**a**) and (**b**) are not differentiably conjugate on a neighborhood of $(1, 2)$.

iii. The Hartman-Grobman Theorem guarantees Topological conjugacy provided the linearizatons are hyperbolic and the number of eigenvalues with real part positive is the same for both. They both have exactly one eigenvalue with postive real part so the Theorem implies topological conjugacy.

iv. The Flow Box Theorem implies that any two systems of the same dimension are differentiably conjugate at non equilibrium points. Since $(2, 2)$ is not an equilibrium for system (**a**) and (**b**), differentiable conjugacy follows.

v. Equilibria are solutions of the equation $F(a, X) = 0$. The Implicit Function Theorem implies that near $a = 0, \underline{X} = (1, 2)$ the solutions are a differentiable curve $X = K(a)$ when $D_X F(0, \underline{X})$ is invertible. This matrix is the coefficient matrix of the linearized equation at $(1, 2)$. Its eigenvalues are $-1, 2$. Since zero is not an eigenvalue, the matrix is invertible.

3. (3+3+4+1+4=15 points). On $\mathbb{R}^2$ consider the system

$$x' = -2x - y^2 := f(x, y), \qquad y' = -2xy - 4y^3 := g(x, y)\,.$$

i. Verify that the neccessary and sufficient condition to be a gradient system is satisfied.

ii. Determine the unique potential $V(x, y)$ with $V(0, 0) = 0$ that generates the system.

iii. Is V a lyapunov function? It is a strict lyapunov function?

iv. What do you conclude about stability?

v. Show that every solution $X(t)$ approaches the origin as $t \to \infty$. Attn. Instruction 4.

Solution. i. The necessary and sufficient condition to be a gradient system is

$$\frac{\partial f}{\partial y} \;=\; \frac{\partial g}{\partial x}\,.$$

Compute

$$\frac{\partial f}{\partial y} = \frac{\partial(-2x - y^2)}{\partial y} = -2y\,, \qquad \frac{\partial g}{\partial x} = \frac{\partial(-2xy - 4y^3)}{\partial x} = -2y\,.$$

The condition is satisfied.

ii. Denote the potential by $V(x, y)$. Must have

$$\frac{\partial V}{\partial x} = -f = 2x + y^2\,.$$

Integrate with respect to x to find

$$V(x, y) \;=\; x^2 + xy^2 + h(y)$$

where h is an arbitrary function of y.

Must have in addition

$$-g = \frac{\partial V}{\partial y} = 2xy + h'(y)\,.$$

This holds if and only if $h' = 4y^3$ if and only if $h = y^4 + C$ for a constant C. Therefore one must have

$$V(x,y) \;=\; x^2 + xy^2 + y^4 + C\,.$$

Evaluating at the origin implies that $C = 0$ so

$$V(x,y) \;=\; x^2 + xy^2 + y^4\,.$$

One can then either verify that V solves or note that the steps in the derivation are all if and only if so that the argument is reversible.

iii. For all gradient systems one has that on orbits

$$\frac{d}{dt} V \;=\; -\|\mathrm{grad}\, V\|^2 \;\le\; 0$$

so V is non increasing on orbits.

To show that V is a lyapunov function need to show that V has a strict minimum at $(0,0)$. Toward that end complete the square to write

$$V(x,y) = \Big(x + \frac{y^2}{2}\Big)^2 - \frac{y^4}{4} + y^4 \;=\; \Big(x + \frac{y^2}{2}\Big)^2 + \frac{3y^4}{4}$$

the sum of two nonegative quantities so $V \ge 0$. In order for V to vanish one must have

$$x + \frac{y^2}{2} = 0 \qquad \text{and} \qquad y = 0\,.$$

Thus $V = 0$ only at the origin which is therefore a strict minimum.

Therefore V is a lyapunov function.

In order to be a strict lyapunov function need to show that $\dot V < 0$ on a punctured disk $0 < x^2 + y^2 < r^2$. That is equivalent to showing that $(f,g) \ne 0$ on $0 < x^2 + y^2 < r^2$. The equation $f = g = 0$ reads

$$2x + y^2 = 0\,, \qquad 2xy + 4y^3 = 0\,.$$

Using the first to eliminate x from the second yields

$$0 = (-y^2)y + 4y^3 = 3y^3$$

whence $y = 0$. Then the first equation implies $x = 0$ so the origin is the unique equilibrium.

Therefore V is a strict lyapunov function.

iv. The existence of a strict lyapunov function implies that the origin is an asymptotically stable equilibrium.

v. Apply the LaSalle principal. Given $X(0)$ define $\alpha := V(X(0))$. Define

$$\mathcal{P} \;:=\; \Big\{(x,y) \;:\; V(x,y) \le \alpha\Big\} \qquad \text{so} \qquad X(0) \in \mathcal{P}\,.$$

Since $V \nearrow \infty$ as $x, y \to \infty$ it follows that $\mathcal{P}$ is bounded. The continuity of V implies that $\mathcal{P}$ is closed. Since the system is gradient like, $\mathcal{P}$ is positively invariant.

To complete the proof that $X(t) \to 0$ using LaSalle's principle, need to show that the only orbit in $\mathcal{P}$ on which V is constant is the equilibrium 0. For V to be constant one would have to have an orbit consisting entirely of equilibria, hence the orbit is an equilibrium. The only equilibrium is 0 so the only such orbit is 0 itself. LaSalle's Theorem applies.

4. (3+3+3+2=11 points). Consider the hard spring equation

$$x'' \;=\; -x^3\,. \tag{1}$$

• Homework 12 writing $3 = 2n + 1$ with $n = 1$ showed that the energy

$$E \;=\; \frac{(x')^2}{2} \;+\; \frac{x^4}{4}$$

is independent of time. In addition, the orbit through $(A, 0)$ in x, x'-space has period that is proportional to $1/A$.

• The last lecture of the course showed using LaSalle that if a friction term $-a\,x'$ is added to the right with $a > 0$ then all solutions converge to the equilibrium.

• The last class showed a computer demonstration of chaos on an attracting set if in addition to friction the system is driven periodically.

The next questions concerns the equation (1) **without damping and without driving**. The orbits are the level curves of energy.

i. Find the intercepts of the level curves of the energy with the $x = 0$ and $x' = 0$ axes.

ii. Roughly sketch the level curves of the energy for $0 < E << 1$ and for $1 << E$.

iii. Is E a lyapunov function for the equilibrium? Is it a strict lyapunov function?

iv. What do you conclude about stability?

Solution. i. The intercepts with $x = 0$ satisfies $E = (x')^2/2$. The intecept in the x, x'-plane are the two points

$$\big(0\,,\,\pm\sqrt{2E}\big)\,.$$

The intercepts with $x' = 0$ satisfy $x^4 = 4E$ so are

$$\big(\pm\,(4E)^{1/4}\,,\,0\big)\,.$$

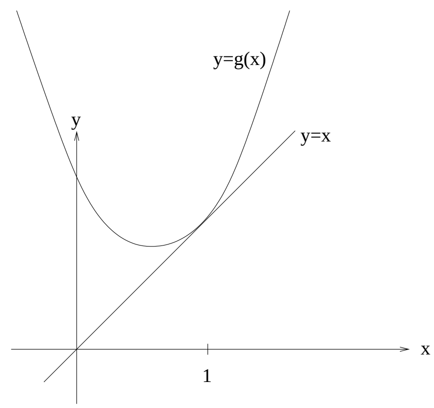
ii. When $0 < E << 1$ the fourth root of E is much larger than the square root so the level curve is a very small and very elongated ovaloid MUCH longer along the x axis. It is convex and orthogonal to both axes.

When $1 << E$ the square root is much larger than the fourth root so the level curve is a very large and very elongated ovaloid MUCH longer along the x' axis. It is convex and orthogonal to both axes.

iii. E is constant on orbits and has a strict minimum at $0, 0$ so it is a lyapunov function but NOT a strict lyapunov function. To be strict it would need to be strictly decreasing on orbits.

iv. The origin is stable by Lyapunov's Theorem. It is not asymtotically stable since the orbit lies on the level curve so does not converge to the origin.

5. (3+4+2+3 =12 points) The convex graph of the function $g(x) := x^4 - 3x + 3$ is sketched below. The graph $y = g(x)$ is tangent to the line $y = x$ at the point $(1, 1)$.



Define the family of maps $f(a, x) := a + g(x)$.

i. For $-1 << a < 0$ how many fixed points does the map f have? What is their approximate position? **Hint.** You need only indicate their rough position with respect to $x = 1$.

ii. (2+2 points) Determine the stability of the fixed point(s). Attention Instructions 3,4.

iii. For $0 < a$, how many fixed points does the map f have?

iv. Determine the stability of the unique fixed point $x = 1$ for $a = 0$.

Solution. **i.** The graph of f for such a is equal to the graph $y = g(x)$ shifted vertically down by $|a|$ units. It crosses $y = x$ at two points, one just to the left of $(1, 1)$ and the other just to the right.

ii. Since $g(x)$ is convex with $g'(1) = 1$ one has $g' < 1$ for $x < 1$ and $g' > 1$ for $x < 1$. Therefore at the crossing to the left the slope is slightly less than one and on the crossing to the right it is slightly greater than one.

It follows by the linearization criterion for stability of fixed points, that the fixed point on the left is stable and the fixed point on the right is unstable.

iii. For $a > 0$ the graph of g is lifted a units and therefore lies a units above $y = x$. There are no fixed points in this case.

iv. For $a = 0$ consider first orbits starting just to the left of 1. For those points $f(0, x) = g(x) > x$ and $g(x) < 1$. The orbit is monotone increasing and to the left of the nearest fixed point 1. The Fundamental Theorem of Monotone Maps implies that the orbit approaches 1.

Consider next orbits starting just to the right of 1. Again one has $g(x) > x$ and the orbit is monotonically increasing. On this orbit $g(x) > 1$ and there is no fixed point to the right of 1. The Fundamental Theorem of Monotone Maps implies that the orbit $x^n \nearrow \infty$ as $n \to \infty$. Therefore the fixed point $x = 1$ is unstable.