Math 558. Fall 2014. Prof. J. Rauch. NAME ....

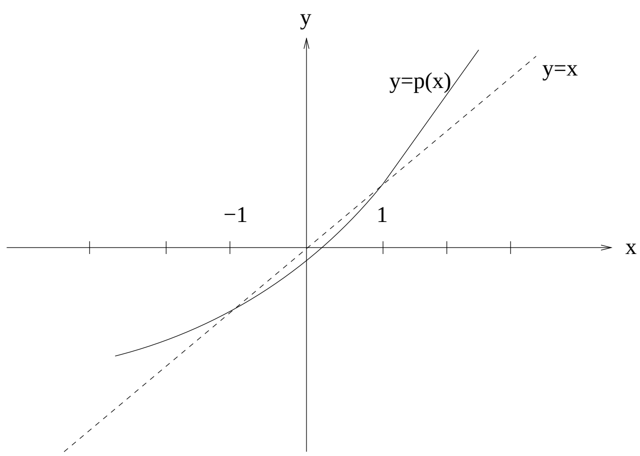
Midterm Exam Solutions, October 16, 2014

Instructions. 1. Two sides of a 3in. $\times$ 5in. sheet of notes from home. Closed book.
2. No electronics, phones, cameras, ... etc.
3. Show work and explain clearly.
4. There are 3 questions.
5. You may use the back of the pages. Extra pages are available.

1. (2+4+5+3+1+5 points). **i.** Find all equilibria of the real scalar differential equation $x' = x^4 - 1$.

ii. Draw the phase line diagram.

iii. Denote by $\phi(t,x)$ the flow and by $p(x) := \phi(1,x)$ the time one map also known as the Poincaré map. In the handouts it is proved that $\partial\phi/\partial x > 0$ for general equations $x' = f(t,x)$. On the axes provided, sketch the rough form of the curve $y = p(x)$ labeling as necessary. The graph should be consistent with the information from i. and ii. Indicate important features.



iv. For $|b| << 1$ explain how the notion of structural stability shows that the equation

$$x' = x^4 - 1 + b\,\sin 2\pi t$$

has exactly two solutions periodic with period equal to 1.

v. When b is very small explain why the periodic solutions are close to the constant functions $x(t) = \pm 1$.

vi. Define $x(t,b)$ to be the solution of the initial value problem with $x(t,b) = 1$. Compute an initial value problem satisfied by the leading term in perturbation theory,

$$x(t,b) = x(t,0) + b\,z(t) + \text{higher order in } b, \qquad z(t) \ := \ \frac{\partial x(t,b)}{\partial b}\bigg|_{b=0}.$$

Solve exactly the initial value problem.

Solution. i. The equilibria are the solutions of $x^4 - 1 = 0$, therefore $x = \pm 1$.

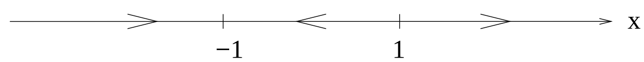
ii. The real line is divided into three intervals, $]-\infty,-1[$, $]-1,1[$, and $]1,\infty[$ by the equilibria. On each interval x^4-1 does not vanish so is of one sign.

On the interval $]-\infty,-1[$, $x^4-1>0$ so solutions move to the right. The sign is easily verified by considering $x\to-\infty$ where $x^4-1\to+\infty$.

On the interval $]-1,1[$, $x^4-1<0$ so solutions move to the left. The sign is easily verified by considering $x=0$ where $x^4-1=-1<0$.

On the interval $]1,\infty[$, $x^4-1>0$ so solutions move to the right. The sign is easily verified by considering $x\to\infty$ where $x^4-1\to+\infty$.

The phase line diagram is as follows.



iii. On the graph note the following features of the graph $y=p(x)$.

The curve $y=p(x)$ is strictly increasing.

The curve $y=p(x)$ intersects the line $y=x$ at exactly the equilibrium points $x=\pm1$.

On the intervals $]-\infty,-1[$ and $]1,\infty[$, $p(x)>x$.

On the interval $]-1,1[$, $p(x)<x$.

Some observed that the theorem in class implies that since the right hand side of the differential equation is convex as a function of x for all times t, the map $p(x)$ is also convex. This was not required.

iv. The curve $y=p(x)$ is strictly increasing and intersects $y=x$ transversally at exactly the two points $x=\pm1$. Under a small perturbation of the differential equation, the curve $y=p(x)$ is changed only a little bit.

When a strictly increasing curve is slightly perturbed it remains strictly increasing.

When the curve that crosses transversally at $x=\pm1$ is slightly perturbed it will still cross $y=x$ transversally at exactly two points, one close to $x=-1$ and a second close to $x=+1$.

This is exactly what happens when $p(x)$ is replaced by the time one map of the time periodic problem with $0\neq b$ and $|b|<<1$.

The intersections with $y=x$ for such a time periodic problem correspond exactly to periodic orbits with period 1. So there are exactly two such periodic orbits one with initial condition close to $x=-1$ and the second with initial condition close to $x=1$.

v. The solution curves coressponding to the fixed point near $x=-1$, starts near $x=-1$. Thus the differential equation and initial value are nearly the same as those for the equilibrium solution -1 of the unperturbed equation. Therefore for $0\le t\le 1$ the solution curves are close. Since both are periodic with period equal to one, they are close for all time.

Similar reasoning applies to the other periodic orbit.

vi. Differentiating the equation with respect to b yields

$$\Big(\frac{\partial^2 x}{\partial t\,\partial b}\Big) \;=\; 4\,x^3\,\frac{\partial x}{\partial b}\;+\;\sin\big(2\,\pi\,t\big)\,.$$

Differentiating the initial condition with respect to b yields

$$\frac{\partial x}{\partial b}\;=\;0\,.$$

Setting $b = 0$ in the preceding two equations yields

$$z' \;=\; 4\,x(t,0)^3\,z + \sin\big(2\pi t)\big)\,, \qquad z(0) \;=\; 0\,.$$

Using the unperturbed solution $x(t,0) = 1$ shows that $z(t)$ satisfies the initial value problem

$$z' \;=\; 4\,z \;+\; \sin 2\pi t\,, \qquad z(0) \;=\; 0\,.$$

Solve using the method of integrating factors,

$$\frac{d}{dt}\; e^{-4t} z \;=\; e^{-4t}\Big(z' - 4z\Big) \;=\; e^{-4t}\,\sin 2\pi t\,.$$

The fundamental theorem of calculus implies that

$$e^{-4t}\,z(t) \;=\; e^{4t} z\Big|_0^t \;=\; \int_0^t \frac{d}{dt}\; e^{-4t} z\; dt \;=\; \int_0^t e^{-4s}\,\sin 2\pi s\; ds\,.$$

So,

$$z(t) \;=\; e^{4t}\,\int_0^t e^{-4s}\,\sin 2\pi s\; ds\,. \tag{1}$$

Alternatively, using the fundamental matrix e^{4t} the formula of varaition of constants for the solution of the inhomogeneous equation with initial value 0 imediately gives (1).

2. (3+3+3+3 points). Consider the family of systems

$$X' \;=\; A\,X\,, \qquad A \;:=\; \begin{pmatrix} 2\alpha & -\beta \\ \beta & 0 \end{pmatrix}$$

with α and β real.

i. Find the region in the $\alpha\beta$-plane where the phase plane is a center.

ii. Find the region in the $\alpha\beta$-plane where the phase plane is an outward spiral.

iii. Find the region in the $\alpha\beta$-plane where the phase plane is a saddle.

iv. Find the region in the $\alpha\beta$-plane where there exist a non constant continuous conserved quantity.

Solution. The eigenvalues are the roots of the quadratic equation

$$0 \;=\; \det\big(A - \lambda I\big) \;=\; \det\begin{pmatrix} 2\alpha - \lambda & -\beta \\ \beta & -\lambda \end{pmatrix} \;=\; (2\alpha - \lambda)(-\lambda) + \beta^2 \;=\; \lambda^2 - 2\alpha\lambda + \beta^2\,.$$

The roots are

$$\frac{2\alpha \pm \sqrt{4\alpha^2 - 4\beta^2}}{2} \;=\; \alpha \pm \sqrt{\alpha^2 - \beta^2}\,.$$

i. The roots are a pair of complex conjugate numbers when $\alpha^2 - \beta^2 < 0$. Equivalently $\alpha^2 < \beta^2$ or $|\alpha| < |\beta|$.

It is a center when the real parts vanish that is $\alpha = 0$. **Answer.** $\big\{\alpha, \beta \;:\; \alpha = 0 \neq \beta\big\}$.

ii. It is an outward spiral when there are two complex roots with positive real part. **Answer.** $\{\alpha, \beta \ : \ 0 < \alpha < |\beta|\}$.

iii. There are two distinct real roots when $\alpha^2 - \beta^2 > 0$. The phase plane is a saddle when this holds and in addition $|\alpha| < \sqrt{\alpha^2 - \beta^2}$ so the roots have opposite signs. *The latter condition never holds so the system is never a saddle.* Both real roots always have the same sign as α.

iv. There is a continuous nonconstant conserved quantity exactly for saddles and centers. The former do not occur. **Answer.** Same as **i**.

3. (4+4+3+2 points). Consider

$$X' \ = \ \begin{pmatrix} 0 & 1 & -1 \\ -2 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix} X \ := \ A\,X\,.$$

You are given the information that the matrix satisfies

$$\det(zI - A) \ = \ (z-1)^2(z-2)\,.$$

and for the eigenvalue 2, $(0, 1, 1)$ is an eigenvector.

i. Find all eigenvectors for the eigenvalue 1. †

ii. Find a basis for the generalized eigenspace corresponding to the eigenvalue 1.

iii. Find three linearly independent solutions $\Phi_j(t)$ of the differential equation.

iv. What is the simple relation between the matrix $\Psi(\mathrm{t})$ whose columns are the Φ_j and the matrix e^{At}? No explanation needed.

Solution. i.

$$A - I \ = \ \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Find its nullspace by row reduction. Subtract the first row from the last and subtracting twice the first row from the second yields

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The nullspace is given by $x_3 = 0 = -x_1 + x_2$. *The eigenvectors are the nonzero multiples of* $(0, 1, 1)$. Since this is a double root the generalized eigenspace is larger than the set of all eigenvectors.

ii. Compute

$$(A - I)^2 \ = \ \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \ = \ \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

The generalized eigenspace is the two dimensional subspace defined by the single equation $x_1 = x_2$.

† By definition eigenvectors are nonzero vectors.

It is convenient to choose the eigenvector as one basis element, $(1,1,0)$. A second basis element is, for example, $(0,0,1)$.

iii. Corresponding to the two eigenvectors one has the solutions

$$\Phi_1(t) \;=\; e^{2t}\,(0,1,1)\,, \qquad \Phi_2(t) \;=\; e^{t}\,(1,1,0)\,.$$

The solution with initial value the other basis element of the generalized eigenspace is

$$\Phi_3(t) \;=\; e^t\Big(I \;+\; (A-I)t\Big)(0,0,1)\,.$$

iv. $e^{At} \;=\; \Psi(t)\,(\Psi(0))^{-1}$.