Math 558 Fall 2014 Prof. J. Rauch
Homework 12 Due Thursday December 4

The first three problems concern the hard nonlinear spring with integer $n \geq 1$,

$$x'' + x^{2n+1} = 0 .$$

1-2. i. Find p so that if $x(t)$ is a solution then so is $a^p x(at)$ for all $a > 0$.

ii. Prove a conservation of energy law. **Hint.** Recall that for the linear spring the energy law comes from the identity

$$x'\big(m\, x'' + k^2\, x\big) = \frac{d}{dt}\left(m\,\frac{\dot{x}^2}{2} + k^2\,\frac{x^2}{2}\right).$$

iii. Denote by $\underline{x}(t)$ the periodic orbit of amplitude equal to 1, that is the solution of with

$$\underline{x}(0) = 1, \qquad \underline{x}'(0) = 0 .$$

Denote by $\underline{T}$ its period. Show that the solution with amplitude $A > 0$ has period equal to $\underline{T}/A^n$. **Hint.** Use **i**. **Discussion.** This is typical of hard springs, that is springs for which the restoring force grows faster than the linear Hooke's law. For hard springs the period decreases with amplitude. For soft springs it increases. For the linear spring the period is independent of amplitude.

iv. Show that all orbits other than the unique equilibrium are periodic. **Hint.** This is proved in the same way that we proved that the orbits of the undamped pendulum with energy less than the energy of the unstable equilibrium are periodic. Start by sketching the potential energy function. **Discussion.** The point of this problem is to revisit that argument to make sure that you understand it. See Chapter 2 Section 12 of V. Arnold's *Ordinary Differential Equations*. There is an electronic version at the UM library.

3. Exercise 3.6 of the Bifurcations handout.

4. 211-212/7d. In addition to the questions proposed sketch the stable and unstable manifolds of all hyperbolic equilibria and the basins of attraction of all stable equilibria. **Hint.** The level set of a quadratic polynomial in x, y is a conic section. The conic is usually an ellipse or hyperbola with center not at the origin.

5. 211/8c,d.

6. 211-212/4. **Hint.** This equation is discussed on page 193.