Math 558 Fall 2014 | Prof. J. Rauch
Homework 6 | **Due Tuesday 21 October 2014**

There is a midterm exam on Thursday October 16.

We will switch to Tuesday due dates so that no homework is due on the exam day. When possible I will move my office hours from Wednesday to Monday to fit better the Tuesday due date. For example I will have office hours Monday 20 October, 2-4 pm.

My apologies. I will be out of town Oct 9 to Oct 15. So I will not have office hours next week. I will have good internet access most of the time and encourage you to email questions that you may have.

1,2. 135/1e,g. Determine whether the origin is stable. Determine whether the origin is asymptotically stable.

3. 135/4. Determine whether the origin is stable. Determine whether the origin is asymptotically stable.

If you have found the general solution

$$c_1\Phi_1(t) \ + \ c_2\Phi_2(t) \ + \ \cdots \ + \ c_N\Phi_N(t)$$

of an $N\times N$ constant coefficient linear system $X' = AX$ then it is easy to find $M(t) = e^{At}$. The matrix valued function M is the unique solution of the initial value problem

$$\frac{dM}{dt} \ = \ A\,M\,, \qquad M(0) \ = \ I\,. \tag{1}$$

On the other hand if you form the $N \times N$ matrix $\Psi(t)$ whose columns are the linearly independent solutions Φ_j one has

$$\frac{d\Psi}{dt} \ = \ A\Psi\,. \tag{2}$$

Invertible matrix solutions of (2) are called **fundamental matrices**. If Ψ is a fundamental matrix then so is $\Psi(t)K$ for any constant invertible $N \times N$ matrix K. Therefore $M(t) := \Psi(t)\,\Psi(0)^{-1}$ is a solution of the initial value problem (1) so must be equal to e^{At}.

How to compute $\exp(tA)$**.** *Find N linearly independent solutions. Form the matrix $\Psi(t)$ whose columns are those solutions. Then*

$$e^{At} \ = \ \Psi(t)\,\Psi(0)^{-1}\,.$$

You can use this strategy to solve the next two problems.

4. For the matrix in 135/1a from the last homework compute in *finite* closed form the exponentials e^{At}. In that assignment you found 3 linearly independent solutions. An infinite series for e^{At} is not an acceptable answer.

5. Consider

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

from the example on page 130 of Hirsch-Smale-Devaney. Write in the form of an integral using the formula of variation of parameters the solution of

$$X' \;=\; A\,X + (\sin t\,,\,0)\,, \qquad X(0) \;=\; 0\,.$$

Remark. The method of variation of parameters is very useful in analysis and not so useful in computing. In this exercise, the integrals can be computed. If the forcing term had been $(e^{t^2}, 0)$ the integrals could not be computed explicitly.

6. 138/13. You must show that $e^{A+B} \neq e^A e^B$.

Discussion. i. In dimension $d \geq 2$, the desired conclusion is valid for nearly all A, B. For example for an open dense set whose complement has zero volume in the $2d^2$ dimensional space of pairs of $d \times d$ matrices.

ii. While it is not true that $AB \neq BA$ implies $e^{A+B} \neq e^A e^B$ it is true that if $AB \neq BA$ then $e^{tA+tB} \neq e^{tA} e^{tB}$ for most t. They are unequal for all sufficiently small non zero t. Since both are real analytic, it follows that in each bounded time interval there are at most a finite set of exceptional times. Proving the first assertion is an alternate strategy for solving the problem.

7. Exercise 4 of the Turing instability handout.