Math 558 Fall 2014 Prof. J. Rauch
Homework 7 Due Tuesday October 28

**1.** 135/1h.

**2.** 184/1i,ii,iii. **Hint.** **iii** is exactly solvable. **i** use phase lines. The example on pg 162 is a model for **ii**.

**3.** 137/9.

**4.** Prove the **Cayley-Hamilton Theorem.** *If $A$ is an $N \times N$ matrix and $p(z) = \det(zI - A)$ is its characteristic polynomial, then $p(A) = 0$.*

**Remarks. i.** Since $A^j$ is an $N \times N$ matrix, $p(A)$ is a sum of such matrices. **ii.** In class the two by two case was proved by approximating by matrices with distinct eigenvalues then passing to the limit. That also works in the $N \times N$ case. **iii.** The suggestion of the next hint is a better proof relying on generalized eigenspaces. **Hint.** Factor

$$p(z) = \Pi_{j=1}^{k}(z - \lambda_j)^{m_j}$$

with distinct $\lambda_j$. Then

$$p(A) = \Pi_{j=1}^{k}(A - \lambda_j I)^{m_j}\,.$$

Use the Spectral Decomposition Theorem to show that $p(A)v = 0$ for each vector $v$ in a generalized eigenspace $X_\mu$. Then represent an arbitrary vector as a sum of these.

**5.** 136/7 (Variant). I will modify part (d) and add a part (e).

**(d)** With $\omega_1, \omega_2$ as in the text explain why all orbits are periodic when $\omega_1/\omega_2$ is rational.

**(e)** Suppose now that $\omega_1/\omega_2$ is irrational. If you plot a typical orbit in the four dimensional phase space for a long time, what do you expect to find? If you plot in a three dimensional section (much easier to do) say in the $x_1, x_2, \dot{x}_1$ three space what do you expect to see? Finally explain why the plot of a two dimensional projection, say on the $x_1, x_2$ plane, will resemble the figures on the top of page 118 of HSD.

**Discussion.** This question should help to resolve the uneasiness of some students with studying the uncoupled oscillators of §6.2. In the small oscillations approximation of mechanical systems in a neighborhood of a strict minimum of potential energy (we will discuss this later), *all systems look like uncoupled oscillators after clever coordinate change.*

**6.** 137/11.

**7.** 186/7. At $a = -1$ means for $a$ on a neighbhorhood of $a = -1$.